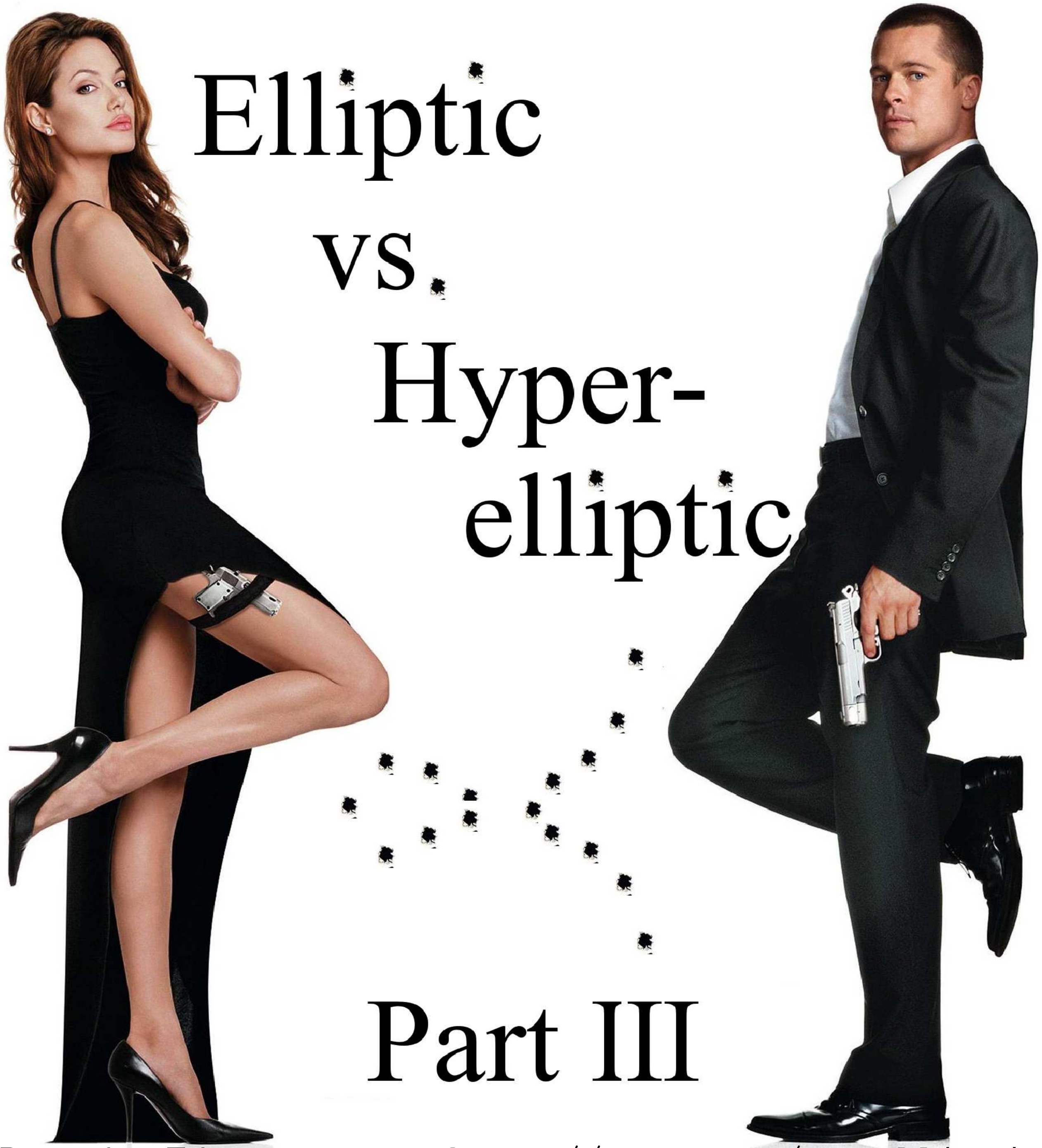
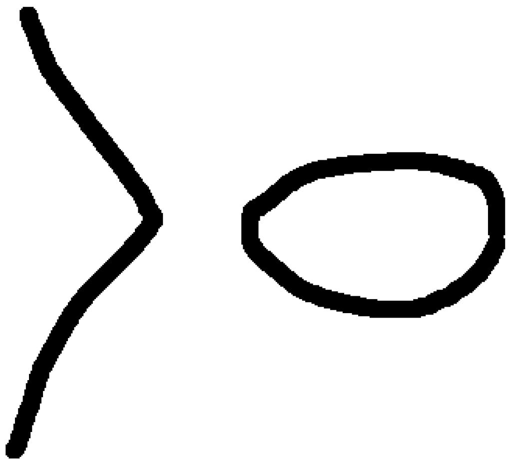
Elliptic vs. Hyper-elliptic

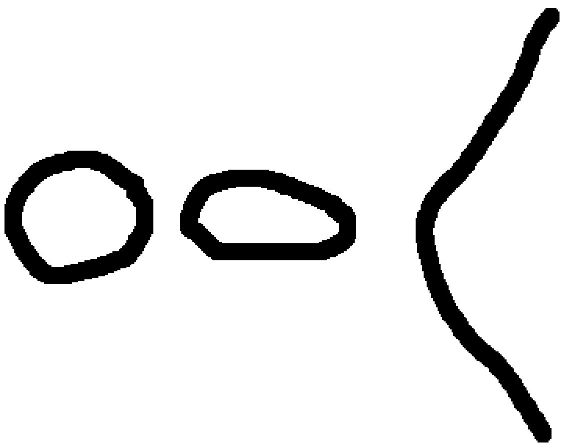
Part III



The Opponents



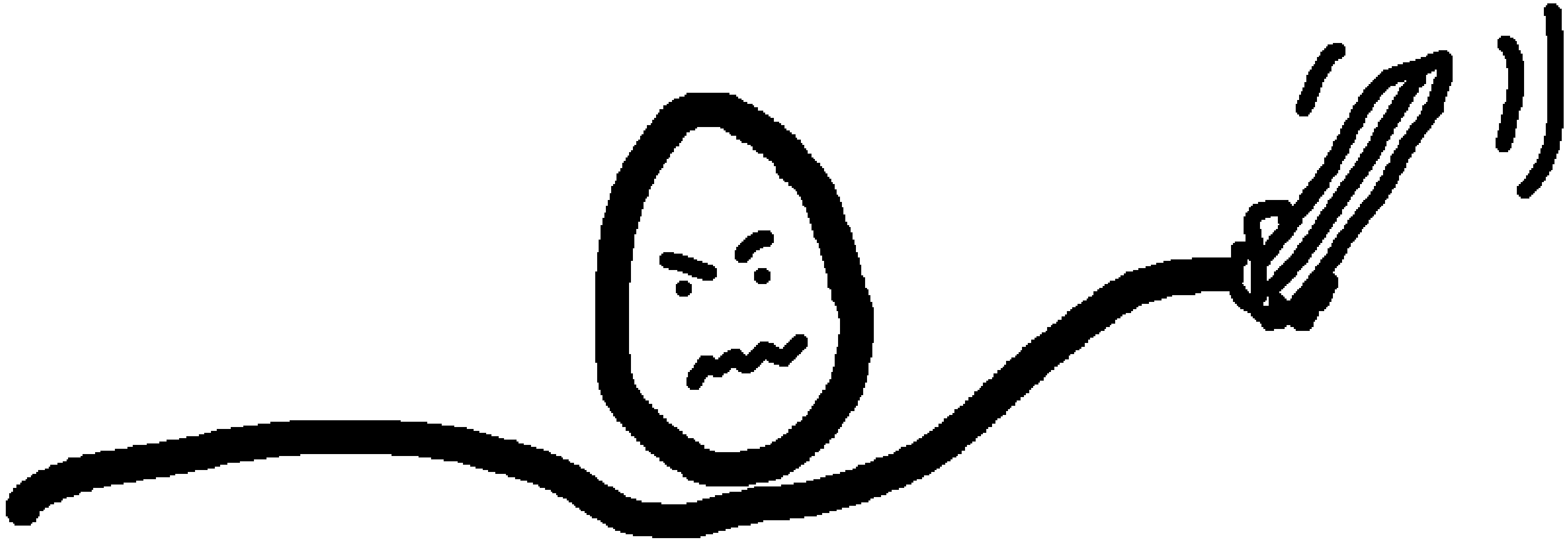
$g = 1$



$g = 2$

(… already after some transformations …)

Elliptic strikes back



Elliptic goes undercover

- Harold M. Edwards Jr., Bulletin of the AMS, electronic April 9, 2007

  $$x^2 + y^2 = a^2(1 + x^2y^2), a^5 \neq a$$

  describes an elliptic curve.
- Edwards shows that generically every elliptic curve can be written in this form – over some extension field.
- Gauss (and this is basically the only mention of this form that Edwards and we could dig out) shows that

  $$x^2 + y^2 = 1 - x^2y^2$$

  is elliptic. To transform this curve we need $\sqrt{-1}$ in the field.

Edwards coordinates

Introduce further parameter and relabel

$$x^2 + y^2 = c^2(1 + dx^2y^2), \; c, d \neq 0, dc^4 \neq 1.$$

- Neutral element is $(0, c)$, this is an affine point!
- $-(x_P, y_P) = (-x_P, y_P)$.
- $P + Q = \left(\frac{x_P y_Q + y_P x_Q}{c(1 + dx_P x_Q y_P y_Q)}, \frac{y_P y_Q - x_P x_Q}{c(1 - dx_P x_Q y_P y_Q)} \right).$

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- $[2]P = \left(\frac{x_P y_P + y_P x_P}{c(1 + d x_P x_P y_P y_P)}, \frac{y_P y_P - x_P x_P}{c(1 - d x_P x_P y_P y_P)}\right).$

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- Unified group operations!

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- $P + Q = \left(\frac{x_P y_Q + y_P x_Q}{c(1 + dx_P x_Q y_P y_Q)}, \frac{y_P y_Q - x_P x_Q}{c(1 - dx_P x_Q y_P y_Q)}\right).$

$$\begin{aligned}
A &= Z_P \cdot Z_Q;\ B = A^2;\ C = X_P \cdot X_Q;\ D = Y_P \cdot Y_Q;\\
E &= (X_P + Y_P) \cdot (X_Q + Y_Q) - C - D;\ F = d \cdot C \cdot D;\\
X_{P \oplus Q} &= A \cdot E \cdot (B - F);\ Y_{P \oplus Q} = A \cdot (D - C) \cdot (B + F);\\
Z_{P \oplus Q} &= c \cdot (B - F) \cdot (B + F).
\end{aligned}$$

Edwards coordinates

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$$\begin{aligned}
A &= Z_P \cdot Z_Q;\ B = A^2;\ C = X_P \cdot X_Q;\ D = Y_P \cdot Y_Q; \\
E &= (X_P + Y_P) \cdot (X_Q + Y_Q) - C - D;\ F = d \cdot C \cdot D; \\
X_{P \oplus Q} &= A \cdot E \cdot (B - F);\ Y_{P \oplus Q} = A \cdot (D - C) \cdot (B + F); \\
Z_{P \oplus Q} &= c \cdot (B - F) \cdot (B + F).
\end{aligned}$$

Needs 10M + 1S + 1C + 1D + 7A. At least one of c, d small.

Fastest unified addition-or-doubling formulae

System	Cost of unified addition-or-doubling
Jacobian	11M+6S+1D; see Brier/Joye '03
Jacobian if $a_4 = -1$	13M+3S; see Brier/Joye '02
Jacobi intersection	13M+2S+1D; see Liardet/Smart '01
Jacobi quartic	10M+3S+3D; see Billet/Joye '01
Hessian	12M; see Joye/Quisquater '01
Edwards ($c = 1$)	10M+1S+1D

- Exactly the same formulae for doubling (no re-arrangement like in Hessian; no if-else)
- No exceptional cases if d is not a square. Formulae correct for all affine inputs (incl. $(0, c), -P$).
- Caveat: Edwards curves have a point of order 2, namely $(0, -c)$.

But wait – there’s more!

How about non-unified doubling?

$$[2]P = \left(\frac{x_P y_P + y_P x_P}{c(1 + d x_P x_P y_P y_P)}, \frac{y_P y_P - x_P x_P}{c(1 - d x_P x_P y_P y_P)}\right)$$

$$= \left(\frac{2 x_P y_P}{c(1 + d(x_P y_P)^2)}, \frac{y_P^2 - x_P^2}{c(1 - d(x_P y_P)^2)}\right)$$

How about non-unified doubling?

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$$= \left(\frac{2 x_P y_P}{c(1 + d(x_P y_P)^2)}, \frac{y_P^2 - x_P^2}{c(1 - d(x_P y_P)^2)}\right)$$

$$= \left(\frac{2 c x_P y_P}{c^2(1 + d(x_P y_P)^2)}, \frac{c(y_P^2 - x_P^2)}{c^2(2 - (1 + d(x_P y_P)^2))}\right)$$

Use curve equation $x^2 + y^2 = c^2(1 + dx^2y^2)$.

How about non-unified doubling?

$$
\begin{aligned}
[2]P &= \left(\frac{x_P y_P + y_P x_P}{c(1 + d x_P x_P y_P y_P)}, \frac{y_P y_P - x_P x_P}{c(1 - d x_P x_P y_P y_P)}\right) \\
&= \left(\frac{2 x_P y_P}{c(1 + d(x_P y_P)^2)}, \frac{y_P^2 - x_P^2}{c(1 - d(x_P y_P)^2)}\right) \\
&= \left(\frac{2c x_P y_P}{c^2(1 + d(x_P y_P)^2)}, \frac{c(y_P^2 - x_P^2)}{c^2(2 - (1 + d(x_P y_P)^2))}\right) \\
&= \left(\frac{2c x_P y_P}{x_P^2 + y_P^2}, \frac{c(y_P^2 - x_P^2)}{2c^2 - (x_P^2 + y_P^2)}\right)
\end{aligned}
$$

How about non-unified doubling?

$$[2]P = \left(\frac{x_P y_P + y_P x_P}{c(1 + d x_P x_P y_P y_P)}, \frac{y_P y_P - x_P x_P}{c(1 - d x_P x_P y_P y_P)}\right)$$

$$= \left(\frac{2 x_P y_P}{c(1 + d(x_P y_P)^2)}, \frac{y_P^2 - x_P^2}{c(1 - d(x_P y_P)^2)}\right)$$

$$= \left(\frac{2 c x_P y_P}{c^2(1 + d(x_P y_P)^2)}, \frac{c(y_P^2 - x_P^2)}{c^2(2 - (1 + d(x_P y_P)^2))}\right)$$

$$= \left(\frac{2 c x_P y_P}{x_P^2 + y_P^2}, \frac{c(y_P^2 - x_P^2)}{2c^2 - (x_P^2 + y_P^2)}\right)$$

Inversion-free version needs 3M + 4S + 3C + 6A.

Can always choose $c = 1$!

Fastest doubling formulae

System	Cost of doubling
Projective	6M+5S+1D; HECC
Hessian	6M+6S; see Joye/Quisquater '01
Jacobi quartic	1M+9S+3D; see Billet/Joye '01
Jacobian	2M+7S+1D; HECC
Jacobian if $a_4 = -3$	3M+5S; see DJB '01
Jacobi intersection	4M+3S+1D; see Liardet/Smart '01
Edwards ($c = 1$)	3M+4S

- Edwards ADD takes 10M+1S+1D, mixed 9M+1S+1D.
- Edwards faster than Jacobian in DBL & ADD.
- Edwards coordinates allow to use windowing methods
- Montgomery takes 5M+4S+1D per bit.

But wait – there's more!

http://cr.yp.to/
newelliptic.html