Edwards Coordinates

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joint work with Daniel J. Bernstein

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Discrete logarithm systems

Cryptographic protocols such as e-voting or digital signatures often apply discrete logarithm systems as a primitive.

The discrete logarithm problem is the problem of finding a secret scalar, namely:

- given a group G generated by some P
- and a *scalar multiple* $Q = [n]P$ of P
- find n (modulo $\ell = \mathrm{ord}(P)$, the order of P).

Within this talk, groups are written additively with group operation $P \oplus Q$ and $[n]P = \underbrace{P \oplus P \oplus \cdots \oplus P}_{n \text{ copies}}$.

Diffie-Hellman Key exchange

Alice		**Bob**
1. secretly generates $a < \vert\langle P\rangle\vert$		1. secretly generates $b < \vert\langle P\rangle\vert$
2. computes $Q_1 = [a]P$		2. computes $Q_2 = [b]P$
3. transmits Q_1		3. transmits Q_2
4. computes $[a]Q_2$	$= [ab]P =$	4. computes $[b] \cdot Q_1$

Common Key: the group element $k = [(ab)]P \in G$.

Proposed groups

- $\mathbb{F}_p^*$, multiplicative group of prime field.
- $\mathbb{F}_q^*$ (same for arbitrary finite field).
- $E(\mathbb{F}_q)$, group of $\mathbb{F}_q$-rational points on elliptic curve E.
- $\mathrm{Pic}^0_C(\mathbb{F}_q)$, divisor class group of curve C over $\mathbb{F}_q$.

Difference in security:

- First two examples have algorithms subexponential in q

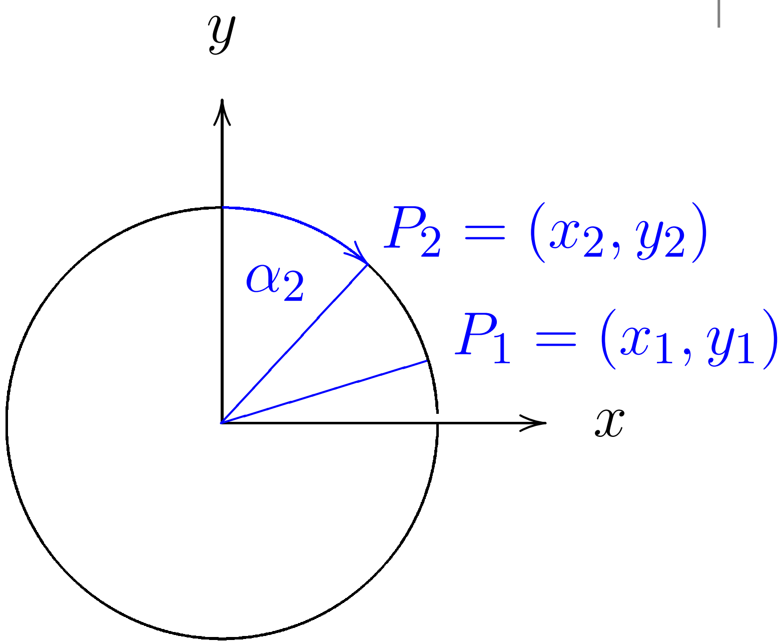
$$O(\exp(c(\log q)^{1/3}(\log\log q)^{2/3})),$$

- for elliptic and small genus hyperelliptic curves the best algorithms are exponential in group size l

$$\Theta(\exp((1/2)\log l)) = \Theta(l^{1/2}).$$

Do you know how to add on a circle?

Let k be a field with $2 \neq 0$.
Circle: $\{(x, y) \in k \times k | x^2 + y^2 = 1\}$



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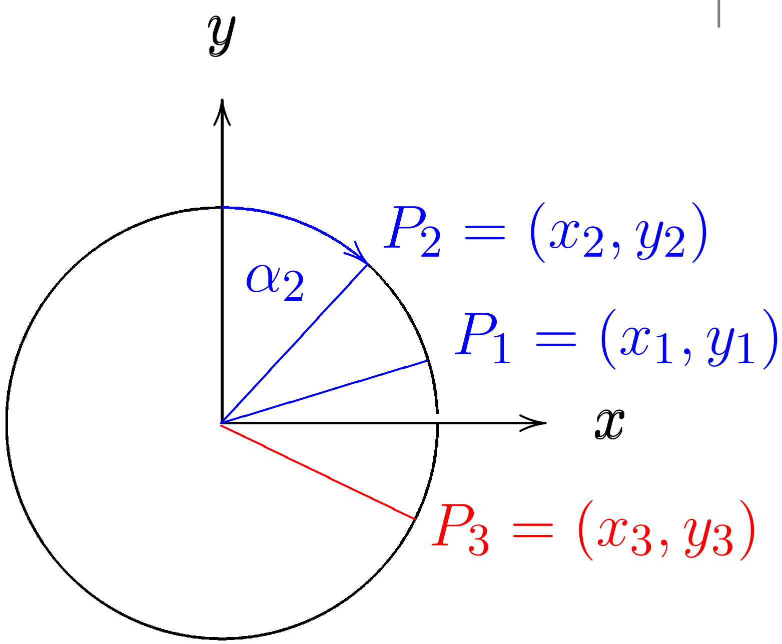
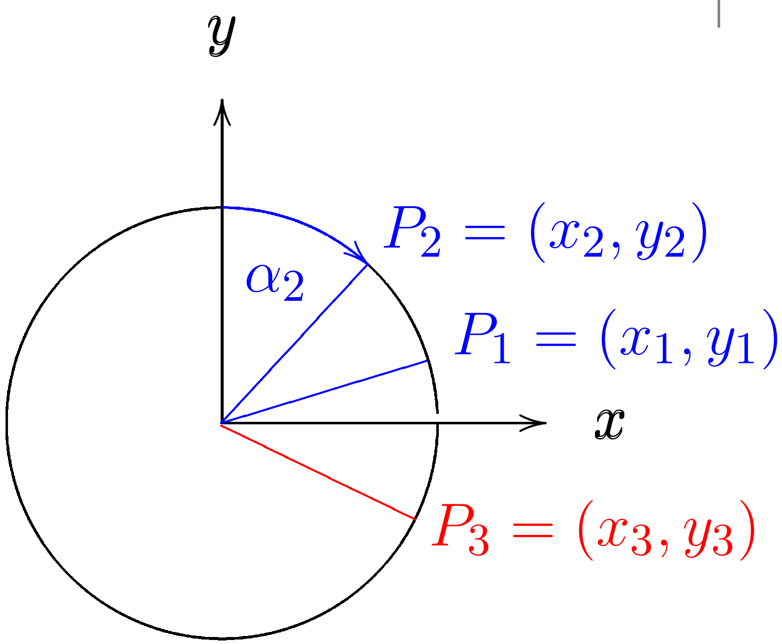


Do you know how to add on a circle?

Let k be a field with $2 \neq 0$.
Circle: $\{(x, y) \in k \times k | x^2 + y^2 = 1\}$
$x_i = \sin(\alpha_i),\ y_i = \cos(\alpha_i)$



$$\begin{aligned} x_3 &= \sin(\alpha_1 + \alpha_2) \\ &= \sin(\alpha_1)\cos(\alpha_2) + \cos(\alpha_1)\sin(\alpha_2) \end{aligned}$$

$$\begin{aligned} y_3 &= \cos(\alpha_1 + \alpha_2) \\ &= \cos(\alpha_1)\cos(\alpha_2) - \sin(\alpha_1)\sin(\alpha_2) \end{aligned}$$

Addition of angles defines commutative group law
$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$, where

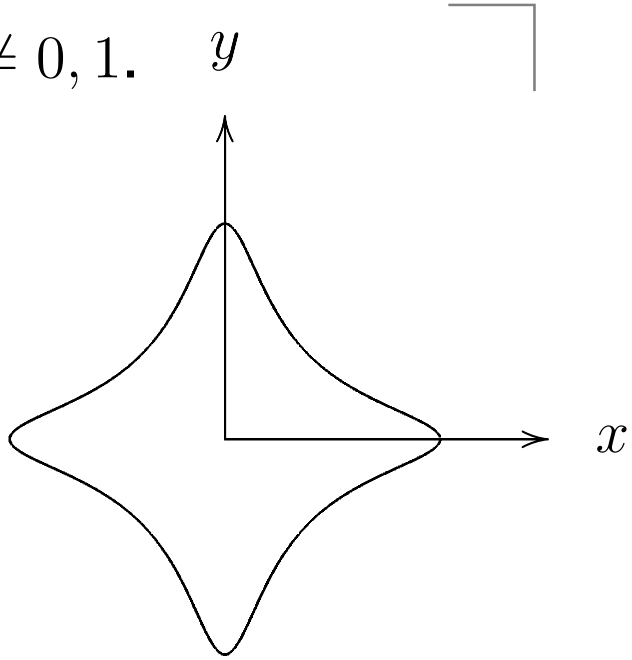
$$x_3 = x_1 y_2 + y_1 x_2 \text{ and } y_3 = y_1 y_2 - x_1 x_2.$$

Now add on an Edwards curve

Let k be a field with $2 \neq 0$. Let $d \in k$ with $d \neq 0, 1$.
Edwards curve:

$$\{(x, y) \in k \times k | x^2 + y^2 = 1 + dx^2y^2\}$$

Harold M. Edwards,
(Bulletin of the AMS, **44**, 393–422, 2007)



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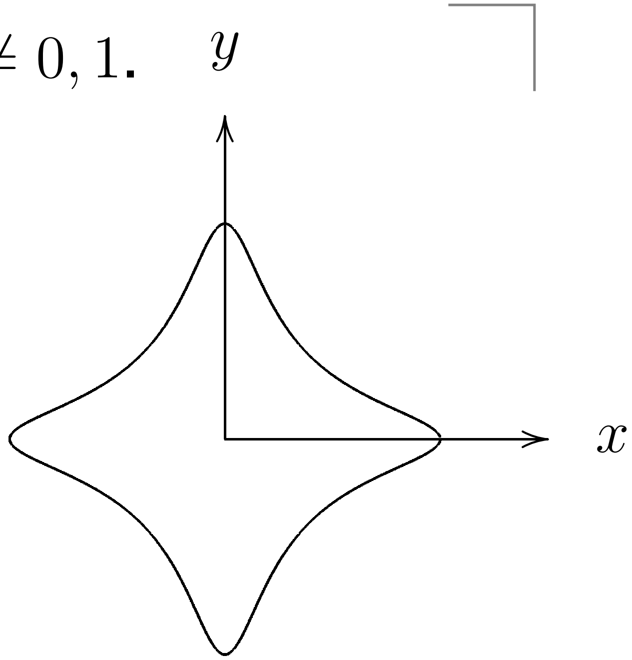

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$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$
defined by Edwards addition law

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2} \text{ and } y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

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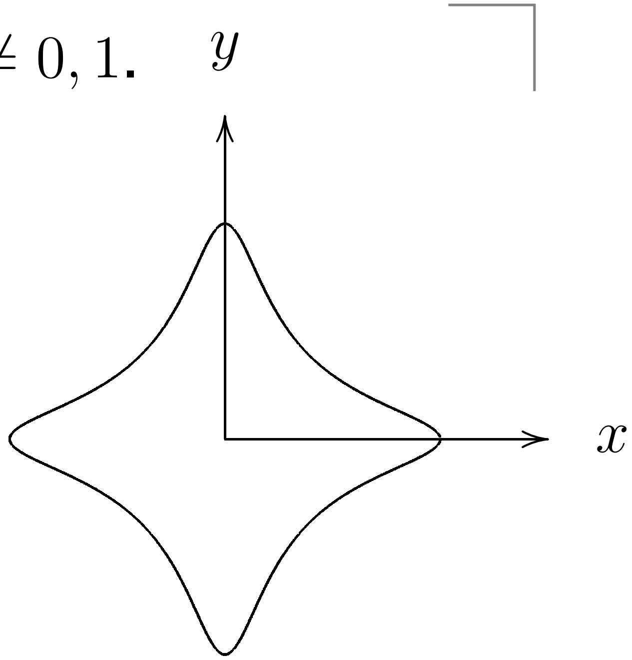

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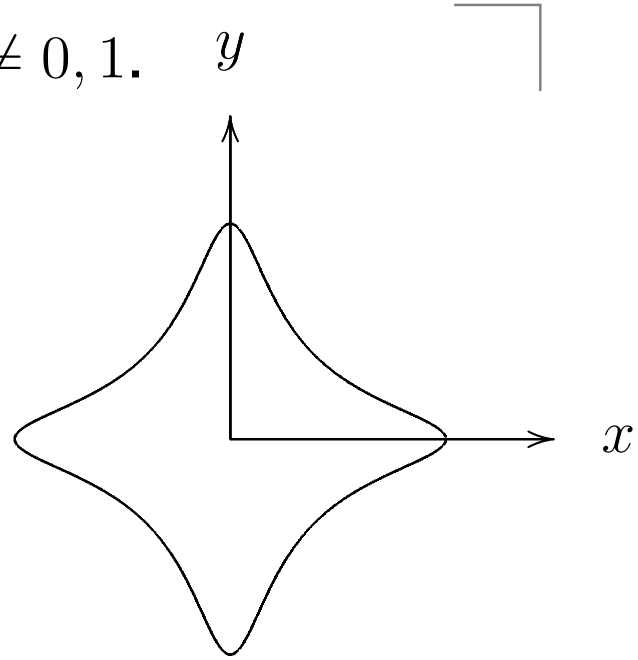

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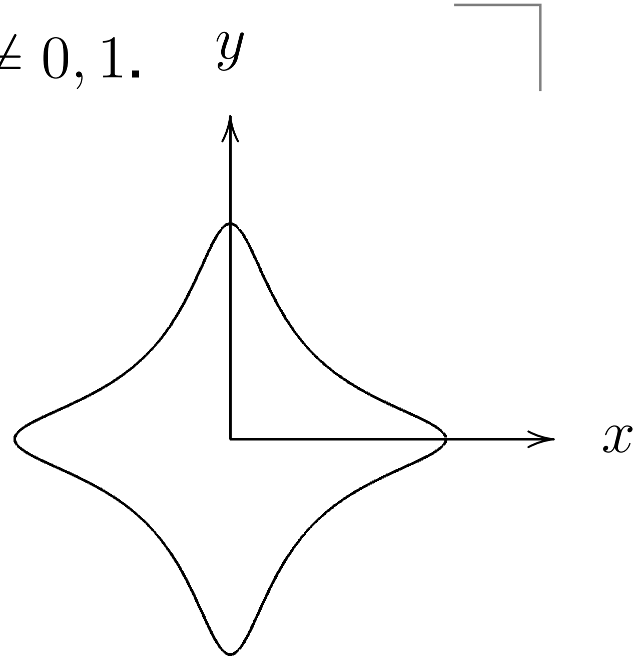

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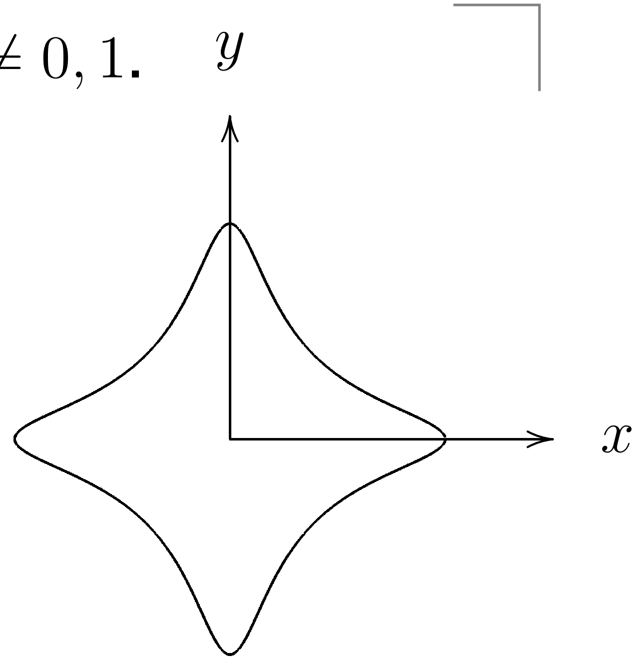

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- Neutral element is $(0, 1)$ (like on circle).
- $-(x_1, y_1) = (-x_1, y_1)$ (like on circle).

Relationship to elliptic curves

- Every elliptic curve with point of order 4 is birationally equivalent to an Edwards curve.
- Let $P_4 = (u_4, v_4)$ have order 4 and shift u s.t. $2P_4 = (0, 0)$. Then Weierstrass form:
$$v^2 = u^3 + (v_4^2/u_4^2 - 2u_4)u^2 + u_4^2 u.$$
- Define $d = 1 - (4u_4^3/v_4^2)$.
- The coordinates $x = v_4 u/(u_4 v),\ y = (u - u_4)/(u + u_4)$ satisfy
$$x^2 + y^2 = 1 + dx^2y^2.$$
- Inverse map $u = u_4(1 + y)/(1 - y),\ v = v_4 u/(u_4 x)$.
- Finitely many exceptional points. Exceptional points have $v(u + u_4) = 0$.

Details on the blow-up

- Points with $v(u+u_4)=0$ on Weierstrass curve map to points at infinity on desingularization of Edwards curve.
- Reminder: $d = 1-(4u_4^3/v_4^2)$.
- $u=-u_4$ is u-coordinate of a point iff

$$\begin{aligned} & (-u_4)^3 + (v_4^2/u_4^2 - 2u_4)(u_4)^2 + u_4^2(u_5) \\ = \ & v_4^2 - 4u_4^3 = v_4^2 d \end{aligned}$$

  is a square, i. e., iff d is a square.
- $v=0$ corresponds to $(0,0)$ which maps to $(0,-1)$ on Edwards curve and to solutions of $u^2 + (v_4^2/u_4^2 - 2u_4)u + u_4^2 = 0$. Discriminant is

$$(v_4^2/u_4^2 - 2u_4)^2 - 4u_4^2 = v_4^4 d,$$

  i. e., points defined over k iff d is a square.

Nice features of the addition law

- Neutral element of addition law is affine point, this avoids special routines (for $(0,1)$ one of the inputs or the result).
- Addition law is symmetric in both inputs.
- $P + Q = \left(\frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2}, \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}\right).$

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- $[2]P = \left(\frac{x_1y_1 + y_1x_1}{1 + dx_1x_1y_1y_1}, \frac{y_1y_1 - x_1x_1}{1 - dx_1x_1y_1y_1} \right).$

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- Unified group operations!
- Having addition law work for doubling removes some checks from the code.

Explicit formulas: addition

- $(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right).$
- Avoid inversions: Use $(X_1 : Y_1 : Z_1)$ with $Z_1 \neq 0$ to represent $(x_1, y_1) = (X_1/Z_1, Y_1/Z_1)$, i. e., $(X_1 : Y_1 : Z_1) = (\lambda X_1 : \lambda Y_1 : \lambda Z_1)$ for $\lambda \neq 0$.
- Addition formulas in projective coordinates:

$$\begin{aligned} A &= Z_1 \cdot Z_2;\ B = A^2;\ C = X_1 \cdot X_2;\ D = Y_1 \cdot Y_2; \\ E &= d \cdot C \cdot D;\ F = B - E;\ G = B + E; \\ X_3 &= A \cdot F \cdot ((X_1 + Y_1) \cdot (X_2 + Y_2) - C - D); \\ Y_3 &= A \cdot G \cdot (D - C); \\ Z_3 &= F \cdot G. \end{aligned}$$

- Needs 10M + 1S + 1D + 7A.

Strongly unified group operations

- Addition formulas work also for doubling.
- E.g. addition law in Weierstrass form $y^2 = x^3 + a_4x + a_6$ is not unified,

$$\lambda = \begin{cases} (y_2 - y_1)/(x_2 - x_1) & \text{if } x_1 \neq x_2, \\ (3x_1^2 + a_4)/(2y_1) & \text{else.} \end{cases}$$

  division by zero if first form used for doubling.
- Help against simple side-channel attacks. Attacker sees uniform sequence of identical group operations, no information on secret scalar given (assuming the field operations are handled appropriately).
- Several systems proposed to deal with this, here comes a brief overview.

Unified Projective coordinates

- Brier, Joye 2002
  Idea: unify how the slope is computed.
- improved in Brier, Déchène, and Joye 2004
- $$\begin{aligned}\lambda &= \frac{(x_1+x_2)^2 - x_1x_2 + a_4 + y_1 - y_2}{y_1+y_2+x_1-x_2} \\ &= \begin{cases} \frac{y_1-y_2}{x_1-x_2} & (x_1,y_1) \neq \pm(x_2,y_2) \\ \frac{3x_1^2+a_4}{2y_1} & (x_1,y_1) = (x_2,y_2) \end{cases}\end{aligned}$$

  Multiply numerator & denominator by $x_1 - x_2$ to see this.
- Proposed formulae can be generalized to projective coordinates.
- Some special cases may occur, but with very low probability, e. g. $x_2 = y_1 + y_2 + x_1$. Alternative equation for this case.

Jacobi intersections

- Chudnovsky and Chudnovsky 1986; Liardet and Smart CHES 2001
- Elliptic curve given as intersection of two quadratics

$$s^2 + c^2 = 1 \text{ and } as^2 + d^2 = 1.$$

- Points $(S : C : D : Z)$ with $(s, c, d) = (S/Z, C/Z, D/Z)$.
- Neutral element is $(0, 1, 1)$.

$$\begin{aligned}
S_3 &= (Z_1C_2 + D_1S_2)(C_1Z_2 + S_1D_2) - Z_1C_2C_1Z_2 - D_1S_2S_1D_2 \\
C_3 &= Z_1C_2C_1Z_2 - D_1S_2S_1D_2 \\
D_3 &= Z_1D_1Z_2D_2 - aS_1C_1S_2C_2 \\
Z_3 &= Z_1C_2^2 + D_1S_2^2.
\end{aligned}$$

- Unified formulas need 13M + 2S + 1D.

Jacobi quartics

- Billet and Joye AAECC 2003

$$E_J : Y^2 = \epsilon X^4 - 2\delta X^2 Z^2 + Z^4.$$

$$\begin{aligned} X_3 &= X_1 Z_1 Y_2 + Y_1 X_2 Z_2 \\ Z_3 &= (Z_1 Z_2)^2 - \epsilon (X_1 X_2)^2 \\ Y_3 &= (Z_3 + 2\epsilon (X_1 X_2)^2)(Y_1 Y_2 - 2\delta X_1 X_2 Z_1 Z_2) + \\ & \quad 2\epsilon X_1 X_2 Z_1 Z_2 (X_1^2 Z_2^2 + Z_1^2 X_2^2). \end{aligned}$$

- Unified formulas need 10M+3S+D+2E
- Can have ϵ or δ small
- Needs point of order 2; for $\epsilon = 1$ the group order is divisible by 4.
- Some recent speed ups due to Duquesne, to Hisil/Carter/Dawson and to Feng/Wu.

Hessian curves

$$E_H : X^3 + Y^3 + Z^3 = cXYZ.$$

Addition: $P \neq \pm Q$	Doubling $P = Q \neq -P$
$X_3 = X_2 Y_1^2 Z_2 - X_1 Y_2^2 Z_1$	$X_3 = Y_1(X_1^3 - Z_1^3)$
$Y_3 = X_1^2 Y_2 Z_2 - X_2^2 Y_1 Z_1$	$Y_3 = X_1(Z_1^3 - Y_1^3)$
$Z_3 = X_2 Y_2 Z_1^2 - X_1 Y_1 Z_2^2$	$Z_3 = Z_1(Y_1^3 - X_1^3)$

- Curves were first suggested for speed
- Joye and Quisquater show

$$[2](X_1 : Y_1 : Z_1) = (Z_1 : X_1 : Y_1) \oplus (Y_1 : Z_1 : X_1)$$

- Unified formulas need 12M.
- Doubling is done by an addition, but not automatically – only unified, not strongly unified.

Fastest unified addition-or-doubling formulae

System	Cost of unified addition-or-doubling
Projective	11M+6S+1D; see Brier/Joye '03
Projective if $a_4 = -1$	13M+3S; see Brier/Joye '02
Jacobi intersection	13M+2S+1D; see Liardet/Smart '01
Jacobi quartic ($\epsilon = 1$)	10M+3S+1D; see Billet/Joye '01
Hessian	12M; see Joye/Quisquater '01
Edwards	10M+1S+1D

- Exactly the same formulae for doubling (no re-arrangement like in Hessian; no if-else)
- See Explicit-Formulas Database (EFD) `hyperelliptic.org/EFD` (also for mixed addition (mADD) and re-additions (reADD)).

Unified addition law

- Strongly unified addition laws indeed remove the check for $P \neq Q$ before addition.
- Some systems allow to omit the check $P \neq -Q$ before addition.
- Most systems still have exceptional cases.
- No surprise:
  "The smallest cardinality of a complete system of addition laws on E equals two."
  (Theorem 1 in Wieb Bosma, Hendrik W. Lenstra, Jr., J. Number Theory **53**, 229–240, 1995)
- Bosma/Lenstra give such system; similar to unified projective coordinates.

Back to Edwards curves

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right)$$

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Intuitive explanation:
The points at infinity (for which the addition law must fail) have minimal field of definition $k(\sqrt{d})$.

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Explicit proof: Let $(x_1, y_1), (x_2, y_2)$ be on curve, i.e., if $x_i^2 + y_i^2 = 1 + dx_i^2 y_i^2$. Write $\epsilon = dx_1 x_2 y_1 y_2$ and suppose $\epsilon \in \{-1, 1\}$. Then $x_1, x_2, y_1, y_2 \neq 0$ and
$dx_1^2 y_1^2 (x_2^2 + y_2^2) = dx_1^2 y_1^2 + d^2 x_1^2 y_1^2 x_2^2 y_2^2$

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$$\begin{aligned} d x_1^2 y_1^2 (x_2^2 + y_2^2) &= d x_1^2 y_1^2 + d^2 x_1^2 y_1^2 x_2^2 y_2^2 \\ &= d x_1^2 y_1^2 + \epsilon^2 \end{aligned}$$

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$$\begin{aligned} d x_1^2 y_1^2 (x_2^2 + y_2^2) &= d x_1^2 y_1^2 + d^2 x_1^2 y_1^2 x_2^2 y_2^2 \\ &= d x_1^2 y_1^2 + \epsilon^2 \\ &= 1 + d x_1^2 y_1^2 = x_1^2 + y_1^2 \end{aligned}$$

Back to Edwards curves

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right)$$

What if denominators are 0?

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Back to Edwards curves

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right)$$

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$$\begin{aligned}(x_1 + \epsilon y_1)^2 &= x_1^2 + y_1^2 + 2\epsilon x_1 y_1 = d x_1^2 y_1^2 (x_2^2 + y_2^2) + 2 x_1 y_1 d x_1 x_2 y_1 y_2 \\ &= d x_1^2 y_1^2 (x_2^2 + 2 x_2 y_2 + y_2^2) = d x_1^2 y_1^2 (x_2 + y_2)^2.\end{aligned}$$

Back to Edwards curves

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right)$$

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$x_2 + y_2 \neq 0 \Rightarrow d = ((x_1 + \epsilon y_1)/x_1 y_1 (x_2 + y_2))^2 \Rightarrow d = \square$

$x_2 - y_2 \neq 0 \Rightarrow d = ((x_1 - \epsilon y_1)/x_1 y_1 (x_2 - y_2))^2 \Rightarrow d = \square$

If $x_2 + y_2 = 0$ and $x_2 - y_2 = 0$ then $x_2 = y_2 = 0$, contradiction.

Complete addition law

- For d not a square in k, the Edwards addition law is complete, there are no exceptional cases.
- Edwards addition law allows omitting all checks
  - Neutral element is affine point on curve.
  - Addition works to add P and P.
  - Addition works to add P and $-P$.
  - Addition just works to add P and any Q.
- Only complete addition law in the literature.
- Bosma/Lenstra strikes over quadratic extension. “Pointless exceptional divisor!”

What if we don't care about uniformity

–

just want best speed for scalar multiplication?

Weierstrass projective Coordinates

$P = (X_1 : Y_1 : Z_1)$, $Q = (X_2 : Y_2 : Z_2)$, $P \oplus Q = (X_3 : Y_3 : Z_3)$
on $E : Y^2Z = X^3 + a_4XZ^2 + a_6Z^3$; $(x, y) \sim (X/Z, Y/Z)$

Addition: $P \neq \pm Q$

$A = Y_2Z_1 - Y_1Z_2, B = X_2Z_1 - X_1Z_2,$

$C = A^2Z_1Z_2 - B^3 - 2B^2X_1Z_2$

$X_3 = BC, Z_3 = B^3Z_1Z_2$

$Y_3 = A(B^2X_1Z_2 - C) - B^3Y_1Z_2,$

Doubling $P = Q \neq -P$

$A = a_4Z_1^2 + 3X_1^2, B = Y_1Z_1,$

$C = X_1Y_1B, D = A^2 - 8C$

$X_3 = 2BD, Z_3 = 8B^3.$

$Y_3 = A(4C - D) - 8Y_1^2B^2$

- No inversion is needed – good for most implementations
- General ADD: 12M+2S
- DBL: 7M+5S
- Fast ... but very different performance of ADD and DBL

Weierstrass Jacobian Coordinates

$P = (X_1 : Y_1 : Z_1)$, $Q = (X_2 : Y_2 : Z_2)$, $P \oplus Q = (X_3 : Y_3 : Z_3)$
on $Y^2 = X^3 + a_4XZ^4 + a_6Z^6$; $(x, y) \sim (X/Z^2, Y/Z^3)$

Addition: $P \neq \pm Q$

$A = X_1Z_2^2, B = X_2Z_1^2, C = Y_1Z_2^3,$
$D = Y_2Z_1^3, E = B - A, F = D - C$
$X_3 = 2(-E^3 - 2AE^2 + F^2)$
$Z_3 = E(Z_1 + Z_2)^2 - Z_1^2 - Z_2^2$
$Y_3 = 2(-CE^3 + F(AE^2 - X_3))$,

Doubling $P = Q \neq -P$

$A = Y_1^2, B = Z_1^2$
$C = 4X_1A, D = 3X_1^2 + a_4B^2$
$X_3 = -2C + D^2$
$Z_3 = (Y_1 + Z_1)^2 - A - B$
$Y_3 = -8A^2 + D(C - X_3)$.

- General ADD: 11M+5S
- mixed ADD ($\mathcal{J} + \mathcal{A} = \mathcal{J}$): 8M+3S
- DBL: 3M+7S (one M by a_4); for $a_4 = -3$: 3M+5S

Chudnovsky Jacobian Coordinates

$P = (X_1 : Y_1 : Z_1 : Z_1^2 : Z_1^3)$, $Q = (X_2 : Y_2 : Z_2 : Z_2^2 : Z_2^3)$,
$P \oplus Q = (X_3 : Y_3 : Z_3 : Z_3^2 : Z_3^3)$ on $Y^2 = X^3 + a_4 X Z^4 + a_6 Z^6$;
$(x, y) \sim (X/Z^2, Y/Z^3)$

Addition: $P \neq \pm Q$

$A = X_1 Z_2^2, B = X_2 Z_1^2, C = Y_1 Z_2^3,$
$D = Y_2 Z_1^3, E = B - A, F = D - C$
$X_3 = 2(-E^3 - 2AE^2 + F^2)$
$Z_3 = E(Z_1 + Z_2)^2 - Z_1^2 - Z_2^2$
$Y_3 = 2(-CE^3 + F(AE^2 - X_3))$,
Z_3^2, Z_3^3,

Doubling $P = Q \neq -P$

$A = Y_1^2,$
$C = 4X_1 A, D = 3X_1^2 + a_4(Z_1^2)^2$
$X_3 = -2C + D^2$
$Z_3 = (Y_1 + Z_1)^2 - A - Z_1^2$
$Y_3 = -8A^2 + D(C - X_3)$
Z_3^2, Z_3^3

- General ADD: 10M+4S
- mixed ADD ($\mathcal{J} + \mathcal{A} = \mathcal{J}$): 8M+3S
- DBL: 3M+7S (one M by a_4)

Montgomery Form

Generalized to arbitrary multiples $[n]P = (X_n : Y_n : Z_n), [m]P = (X_m : Y_m : Z_m)$ with known difference $[m-n]P$ on

$$E_M : By^2 = x^3 + Ax^2 + x$$

Addition: $n \neq m$

$$\begin{aligned} X_{m+n} &= Z_{m-n}\big((X_m - Z_m)(X_n + Z_n) + (X_m + Z_m)(X_n - Z_n)\big)^2, \\ Z_{m+n} &= X_{m-n}\big((X_m - Z_m)(X_n + Z_n) - (X_m + Z_m)(X_n - Z_n)\big)^2 \end{aligned}$$

Doubling: $n = m$

$$\begin{aligned} 4X_nZ_n &= (X_n + Z_n)^2 - (X_n - Z_n)^2, \\ X_{2n} &= (X_n + Z_n)^2(X_n - Z_n)^2, \\ Z_{2n} &= 4X_nZ_n\big((X_n - Z_n)^2 + \big((A+2)/4\big)(4X_nZ_n)\big). \end{aligned}$$

An addition takes 4M and 2S whereas a doubling needs only 3M and 2S. Order is divisible by 4.

Edwards explicit formulas: doubling

- $$(x_1, y_1) + (x_1, y_1) = \left(\frac{x_1y_1 + y_1x_1}{1 + dx_1x_1y_1y_1}, \frac{y_1y_1 - x_1x_1}{1 - dx_1x_1y_1y_1}\right)$$
$$= \left(\frac{2x_1y_1}{1 + d(x_1y_1)^2}, \frac{y_1^2 - x_1^2}{1 - d(x_1y_1)^2}\right)$$

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  $$= \left(\frac{2x_1y_1}{1 + d(x_1y_1)^2}, \frac{y_1^2 - x_1^2}{1 - d(x_1y_1)^2}\right)$$

  Use curve equation $x^2 + y^2 = 1 + dx^2y^2$.

Edwards explicit formulas: doubling

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$$= \left(\frac{2 x_1 y_1}{1 + d (x_1 y_1)^2}, \frac{y_1^2 - x_1^2}{1 - d (x_1 y_1)^2}\right)$$
$$= \left(\frac{2 x_1 y_1}{x_1^2 + y_1^2}, \frac{y_1^2 - x_1^2}{2 - (x_1^2 + y_1^2)}\right)$$

Edwards explicit formulas: doubling

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  $$= \left(\frac{2x_1y_1}{1 + d(x_1y_1)^2}, \frac{y_1^2 - x_1^2}{1 - d(x_1y_1)^2}\right)$$
  $$= \left(\frac{2x_1y_1}{x_1^2 + y_1^2}, \frac{y_1^2 - x_1^2}{2 - (x_1^2 + y_1^2)}\right)$$
- Doubling formulas in projective coordinates:
  $$\begin{aligned} B &= (X_1 + Y_1)^2;\ C = X_1^2;\ D = Y_1^2; \\ E &= C + D;\ H = Z_1^2;\ J = E - 2H; \\ X_3 &= (B - E) \cdot J;\ Y_3 = E \cdot (C - D);\ Z_3 = E \cdot J. \end{aligned}$$
- Needs 3M + 4S + 6A.

Doubling speed overview

System	Cost of doubling (as of today)
Projective	5M+6S+1D; EFD
Projective if $a_4 = -3$	7M+3S; EFD
Hessian	7M+1S; see Hisil/Carter/Dawson '07
Doche/Icart/Kohel-3	2M+7S+2D; see B./Birkner/L./Peters
Jacobian	1M+8S+1D; EFD
Jacobian if $a_4 = -3$	3M+5S; see DJB '01
Jacobi quartic	2M+6S+1D; see EFD
Ext. Jacobi quartic	3M+4S; see Hisil/Carter/Dawson '07
Jacobi intersection	3M+4S; EFD
Edwards	3M+4S;
Doche/Icart/Kohel-2	2M+5S+2D; EFD

- Edwards fastest for general curves, no D.

Addition speed overview

System	Cost of addition
Doche/Icart/Kohel-2	12M+5S+1D; EFD
Doche/Icart/Kohel-3	11M+6S+1D; see B./Birkner/L./Peters '07
Jacobian	11M+5S; EFD
Jacobi intersection	13M+2S+1D; see Liardet/Smart '01
Projective	12M+2S; see Chudnovsky/Chudnovsky '86
Jacobi quartic	10M+3S+1D; see Billet/Joye '03
Hessian	12M; see Sylvester (1800's)
Edwards	10M+1S+1D
Ext. Jacobi quartic	8M+3S+1D; EFD (based on Duquesne)

OOPS!

Inverted Edwards coordinates

- Using the representation $(X_1 : Y_1 : Z_1)$ for the affine point $(Z_1/X_1, Z_1/Y_1)$ ($X_1Y_1Z_1 \neq 0$) gives operation counts:
  - Doubling takes 3M+4S+1D.
  - Addition takes 9M+1S+1D.
- This saves 1M for each addition compared to standard Edwards coordinates.
- Doubling slower by 1D; so choose small d.
- Extended Jacobi quartics need 8M+3S+1D to add.
- Inverted Edwards coordinates are strongly unified system – but not complete.
  Additions involving $(0, \pm 1)$ or $(\pm 1, 0)$ need special handling.
- For details see proceedings.

Addition speed overview

System	Cost of addition
Doche/Icart/Kohel-2	12M+5S+1D; EFD
Doche/Icart/Kohel-3	11M+6S+1D; see B./Birkner/L./Peters ’07
Jacobian	11M+5S; EFD
Jacobi intersection	13M+2S+1D; see Liardet/Smart ’01
Projective	12M+2S; see Chudnovsky/Chudnovsky ’86
Jacobi quartic	10M+3S+1D; see Billet/Joye ’03
Hessian	12M; see Sylvester (1800’s)
Edwards	10M+1S+1D
Ext. Jacobi quartic	8M+3S+1D; EFD (based on Duquesne)
Inverted Edwards	9M+1S+1D; see B./L. ’07

- New speed leader: inverted Edwards.

Single-scalar multiplication using NAF

System	1 DBL, 1/3 mADD
Projective	8M+6.67S+1D
Projective if $a_4 = -3$	10M+3.67S
Hessian	10.3M+1S
Doche/Icart/Kohel-3	4.33M+8.33S+2.33D
Jacobian	3.33M+9.33S+1D
Jacobian if $a_4 = -3$	5.33M+6.33S
Jacobi intersection	6.67M+4.67S+0.333D
Jacobi quartic	4.67M+7S+2.33D
Doche/Icart/Kohel-2	4.67M+6.33S+2.33D
Edwards	6M+4.33S+0.333D

For comparison: Montgomery arithmetic takes 5M+4S+1D per bit.

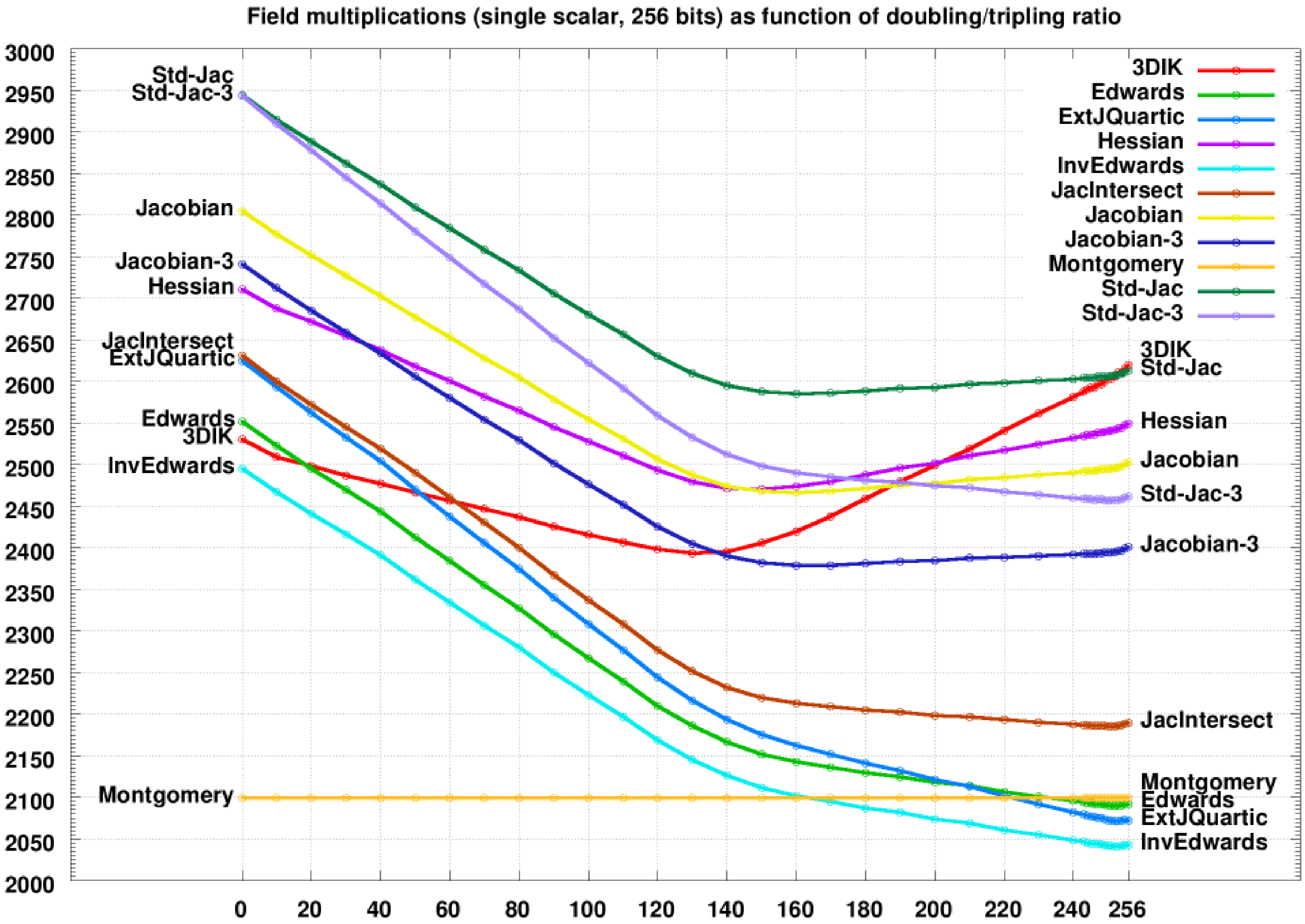
Signed width-4 sliding windows

These counts include the precomputations.

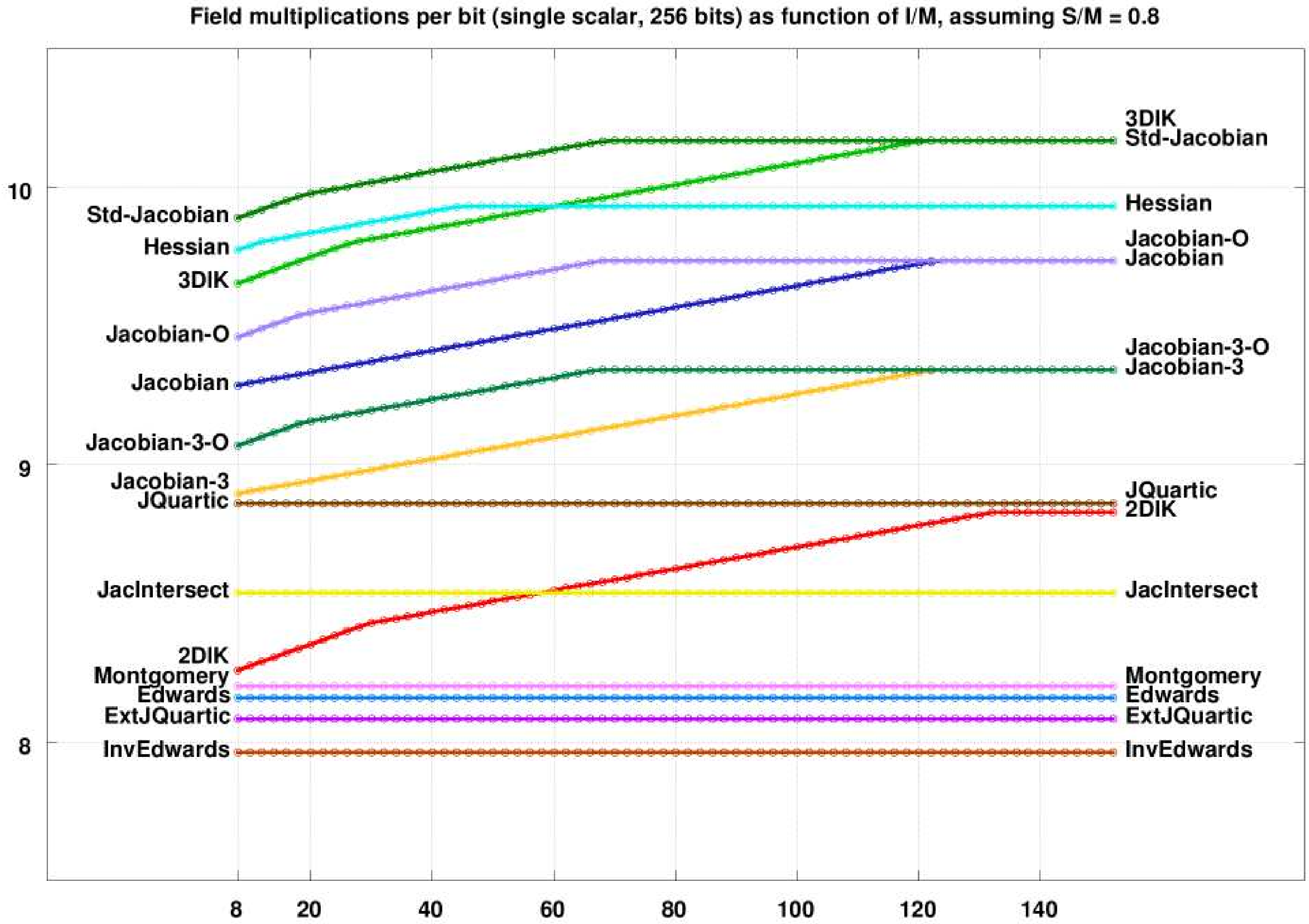
System	0.98 DBL, 0.17 reADD, 0.025 mADD, 0.0035 ADD
Projective	7.17M+6.28S+0.982D
Projective if $a_4 = -3$	9.13M+3.34S
Doche/Icart/Kohel-3	3.84M+7.99S+2.16D
Hessian	9.16M+0.982S
Jacobian	2.85M+8.64S+0.982D
Jacobian if $a_4 = -3$	4.82M+5.69S
Doche/Icart/Kohel-2	4.2M+5.86S+2.16D
Jacobi quartic	3.69M+6.48S+2.16D
Jacobi intersection	5.09M+4.32S+0.194D
Edwards	4.86M+4.12S+0.194D

Montgomery takes 5M+4S+1D per bit. Edwards solidly faster!

Influence of triplings, Indocrypt'07



Influence of inversions, Fq8 2007



Edwards everywhere

- Edwards for SSCA (fastest unified addition).

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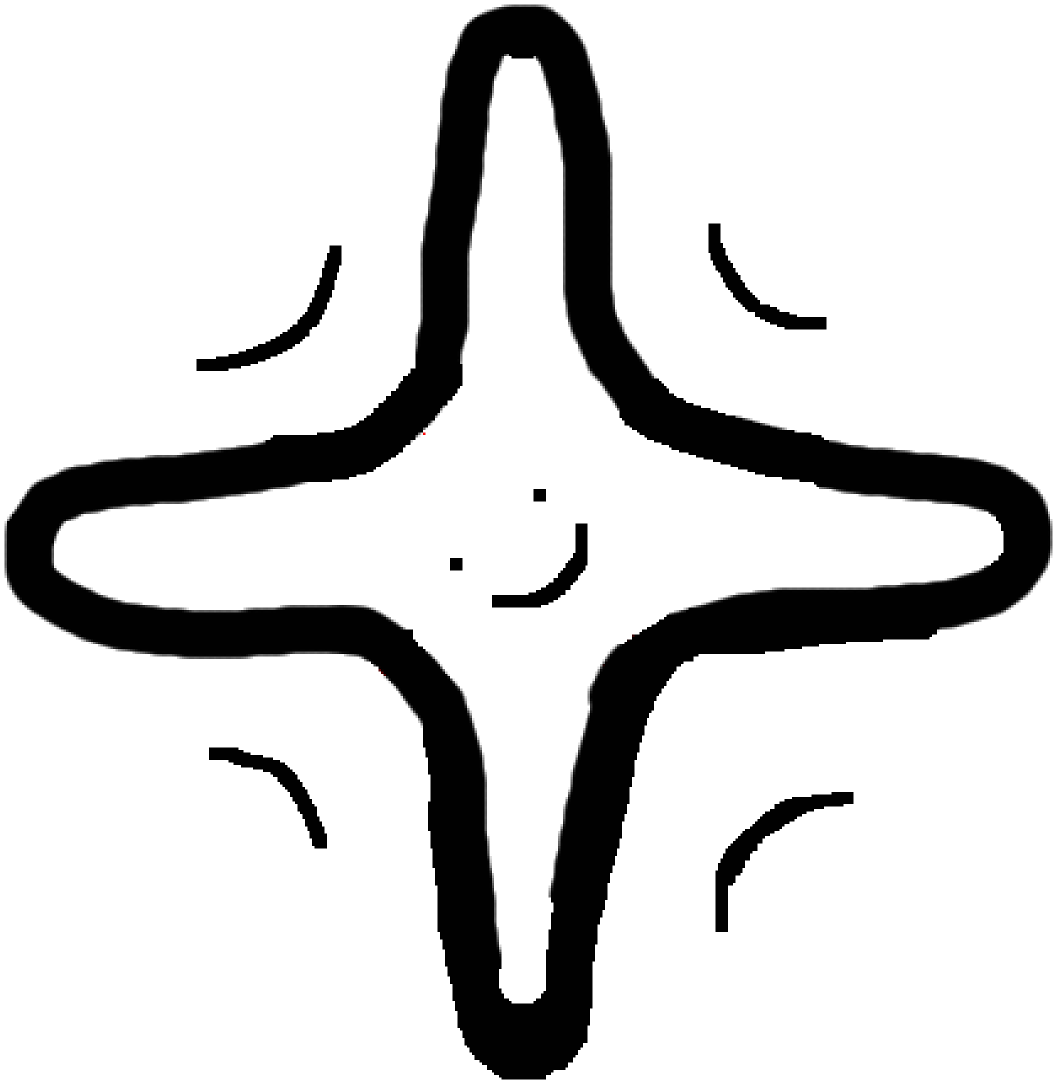
Thank you for your attention!



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