Arithmetic of hyperelliptic curves over finite fields

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28.06.2006

Overview

- Arithmetic on elliptic curves.
- Arithmetic on hyperelliptic curves.
- Montgomery arithmetic

Arithmetic on elliptic curves

Elliptic curve

$$E : y^2 + \underbrace{(a_1x + a_3)}_{h(x)}y = \underbrace{x^3 + a_2x^2 + a_4x + a_6}_{f(x)},\ h, f \in \mathbb{F}_q[x].$$

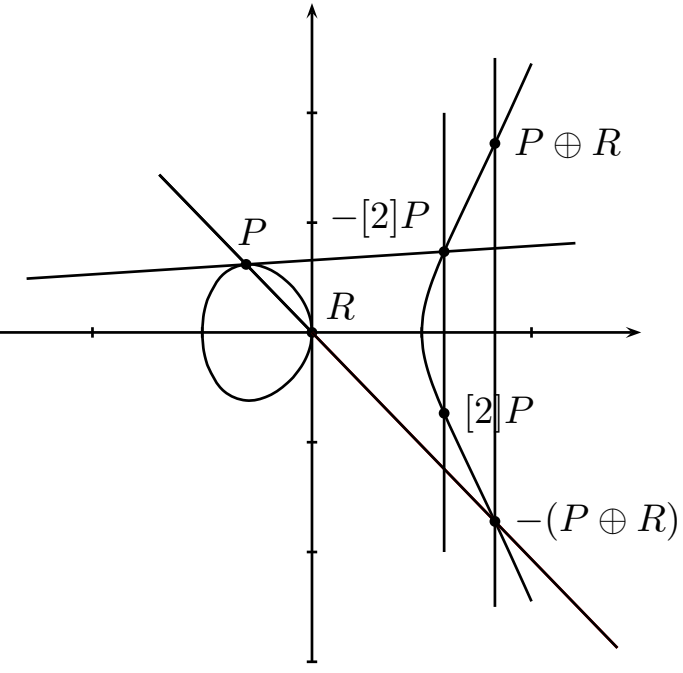
Group: $E(\mathbb{F}_q) = \{\,(x, y) \in \mathbb{F}_q^2 \,:\, y^2 + h(x)y = f(x)\,\} \cup \{\,P_\infty\,\}$

Often $q = 2^r$ or $q = p$, prime. Isomorphic transformations lead to

$$y^2 = f(x), \qquad q \text{ odd},$$
$$y^2 + xy = x^3 + ax^2 + b, \qquad q = 2^r, \text{ curve non-supersingular}$$

Group Law (q odd)

$$E : y^2 = x^3 + a_4x + a_6,\ a_i \in \mathbb{F}_q$$



for $(x_1, y_1) \neq (x_2, -y_2)$:

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= \\ &= (x_3, y_3) = \\ &= (\lambda^2 - x_1 - x_2, \lambda(x_1 - x_3) - y_1),\end{aligned}$$

where

$$\lambda = \begin{cases} (y_2 - y_1)/(x_2 - x_1) & \text{if } x_1 \neq x_2, \\ (3x_1^2 + a_4)/(2y_1) & \text{else.} \end{cases}$$

Addition: 1 I, 2M, 1S
Doubling: 1 I, 2M, 2S.

Projective Coordinates

$P = (X_1 : Y_1 : Z_1)$, $Q = (X_2 : Y_2 : Z_2)$, $P \oplus Q = (X_3 : Y_3 : Z_3)$ on

$$E : Y^2Z = X^3 + a_4XZ^2 + a_6Z^3$$

Correspondence of $(X : Y : Z)$ to $(X/Z, Y/Z)$.

Addition: $P \neq \pm Q$

$A = Y_2Z_1 - Y_1Z_2, B = X_2Z_1 - X_1Z_2,$
$C = A^2Z_1Z_2 - B^3 - 2B^2X_1Z_2$
$X_3 = BC, Z_3 = B^3Z_1Z_2$
$Y_3 = A(B^2X_1Z_2 - C) - B^3Y_1Z_2,$

Doubling $P = Q \neq -P$

$A = a_4Z_1^2 + 3X_1^2, B = Y_1Z_1,$
$C = X_1Y_1B, D = A^2 - 8C$
$X_3 = 2BD, Z_3 = 8B^3.$
$Y_3 = A(4C - D) - 8Y_1^2B^2$

No inversion is needed and the computation times are $12M + 2S$ for a general addition and $7M + 5S$ for a doubling.

Jacobian Coordinates

$P = (X_1 : Y_1 : Z_1)$, $Q = (X_2 : Y_2 : Z_2)$, $P \oplus Q = (X_3 : Y_3 : Z_3)$ on

$$Y^2 = X^3 + a_4 X Z^4 + a_6 Z^6$$

Correspondence of $(X : Y : Z)$ to $(X/Z^2, Y/Z^3)$ (weighted).

Addition: $P \neq \pm Q$

$A = X_1 Z_2^2, B = X_2 Z_1^2, C = Y_1 Z_2^3,$
$D = Y_2 Z_1^3, E = B - A, F = D - C$
$X_3 = -E^3 - 2AE^2 + F^2, Z_3 = Z_1 Z_2 E,$
$Y_3 = -CE^3 + F(AE^2 - X_3),$

Doubling $P = Q \neq -P$

$A = 4X_1 Y_1^2, B = 3X_1^2 + a_4 Z_1^4$
$X_3 = -2A + B^2, Z_3 = 2Y_1 Z_1$
$Y_3 = -8Y_1^4 + B(A - X_3).$

No inversion is needed and the computation times are $12M + 4S$ for a general addition and $4M + 6S$ for a doubling.

Different coordinate systems $y^2 = x^3 + ax + b$

system	points	correspondence
affine ($\mathcal{A}$)	(x, y)	
projective ($\mathcal{P}$)	(X, Y, Z)	$(X/Z, Y/Z)$
jacobian ($\mathcal{J}$)	(X, Y, Z)	$(X/Z^2, Y/Z^3)$
Chudnovsky jacobian ($\mathcal{J}^C$)	(X, Y, Z, Z^2, Z^3)	$(X/Z^2, Y/Z^3)$
modified jacobian ($\mathcal{J}^m$)	(X, Y, Z, aZ^4)	$(X/Z^2, Y/Z^3)$

system	addition			doubling		
affine ($\mathcal{A}$)	2M	1S	1I	2M	2S	1I
projective ($\mathcal{P}$)	12M	2S	–	7M	5S	–
jacobian ($\mathcal{J}$)	12M	4S	–	4M	6S	–
Chudnovsky jacobian ($\mathcal{J}^C$)	11M	3S	–	5M	6S	–
modified jacobian ($\mathcal{J}^m$)	13M	6S	–	4M	4S	–

Arithmetic on hyperelliptic curves

Hyperelliptic Curves

Affine equation of hyperelliptic curve of genus g (with $\mathbb{F}_q$-rational Weierstraß-point at infinity)

$$C : y^2 + h(x)y = f(x)$$

$h(x), f(x) \in \mathbb{F}_q[x]$, f monic, $\deg f = 2g + 1, \deg h \leq g$
non singular, i. e. not both partial derivatives
($2y + h(x)$ and $h'(x)y - f'(x)$)
vanish in any in $(a, b) \in C/\overline{\mathbb{F}_q}$

Examples

Concerning the arithmetic properties one can consider elliptic curves as hyperelliptic curves, i. e.

$$y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6$$

is considered as curve of genus one.

Curve of genus 2 over field of odd characteristic

$$y^2 = x^5 + f_3x^3 + f_2x^2 + f_1x + f_0,$$

provided $f(x)$ has no multiple roots.

Group

Use **Divisors**, i. e. finite sums of points (elements of the free abelian group), of degree zero

$$\sum_{P \in C(\overline{\mathbb{F}_q})} n_P P,\ n_P \in \mathbb{Z}$$

with $\sum_{P \in C(\overline{\mathbb{F}_q})} n_P = 0$ a $n_P = 0$ for almost all P.
One associates the divisor

$$\mathrm{div}(F) = \sum_{P \in C(\overline{\mathbb{F}_q})} v_P(F) P$$

to the function $F \in \mathbb{F}_q(C)$, where v_P is a valuation in the point P. Such divisors are called *principal divisors* Princ_C.
One can show that they have degree zero.

Divisor class group

Consider the factor group of the group of divisors of degree zero Div_C^0 modulo the principal divisors. This way one constructs the divisor class group of degree zero.

$$\mathrm{Pic}_C^0 = \mathrm{Div}_C^0 / \mathrm{Princ}_C.$$

The $\mathbb{F}_q$-rational group elements $\mathrm{Pic}_C^0(\mathbb{F}_q)$ are those which remain fixed under $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

This construction provides an isomorphism between $\mathrm{Pic}_C^0(\mathbb{F}_q)$ and the $\mathbb{F}_q$-rational points of the Jacobian of C.

Representation of group elements

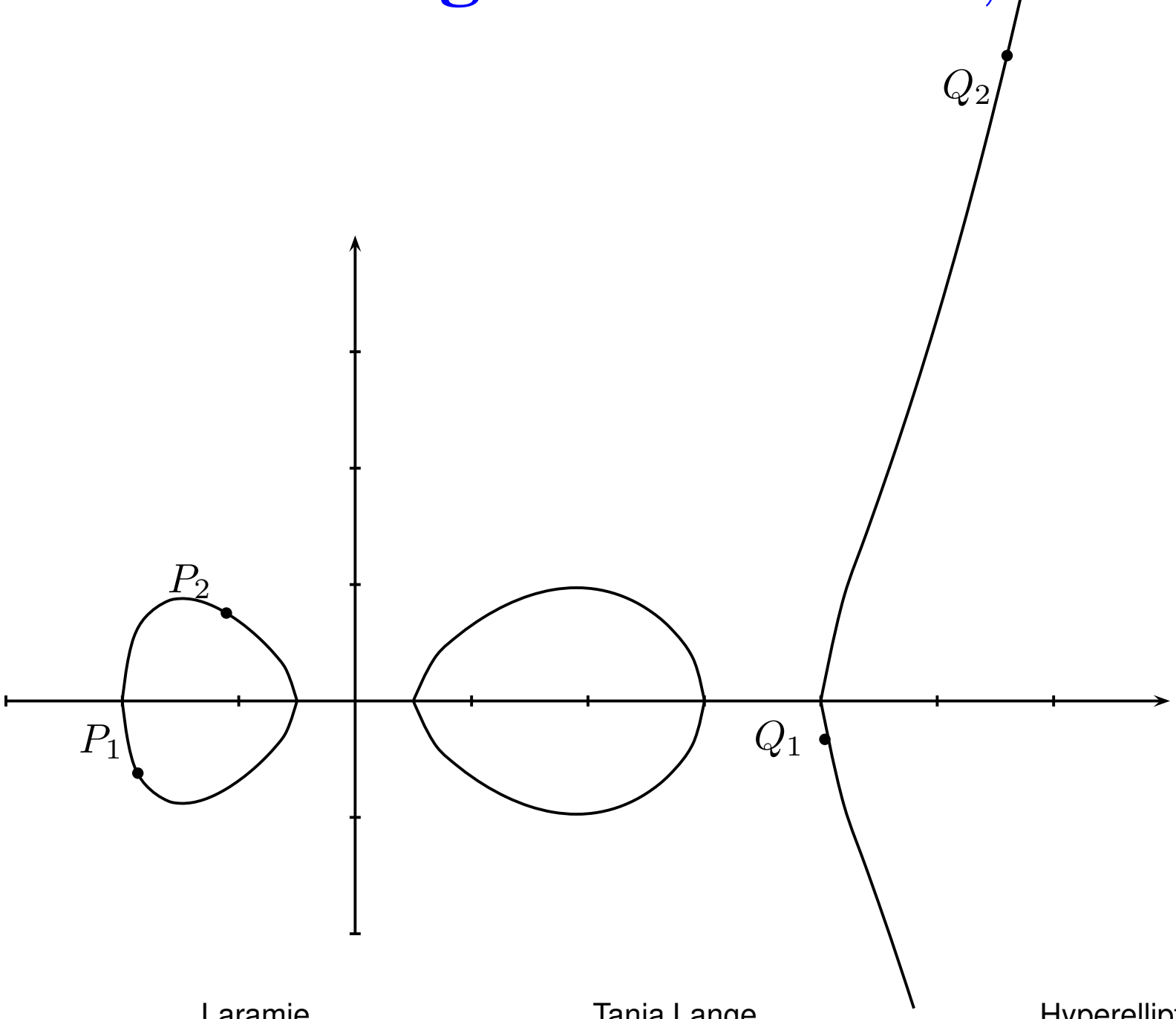
Riemann-Roch allows to find a unique reduced representation by means of a divisor of degree zero with $m \leq g$

$$\bar{D} = \sum_{\substack{i=1 \\ P_i \in C(\overline{\mathbb{F}_q}) \setminus \{P_\infty\}}}^{m} P_i - mP_\infty$$

and $P_i \neq -P_j$ for $i \neq j$.

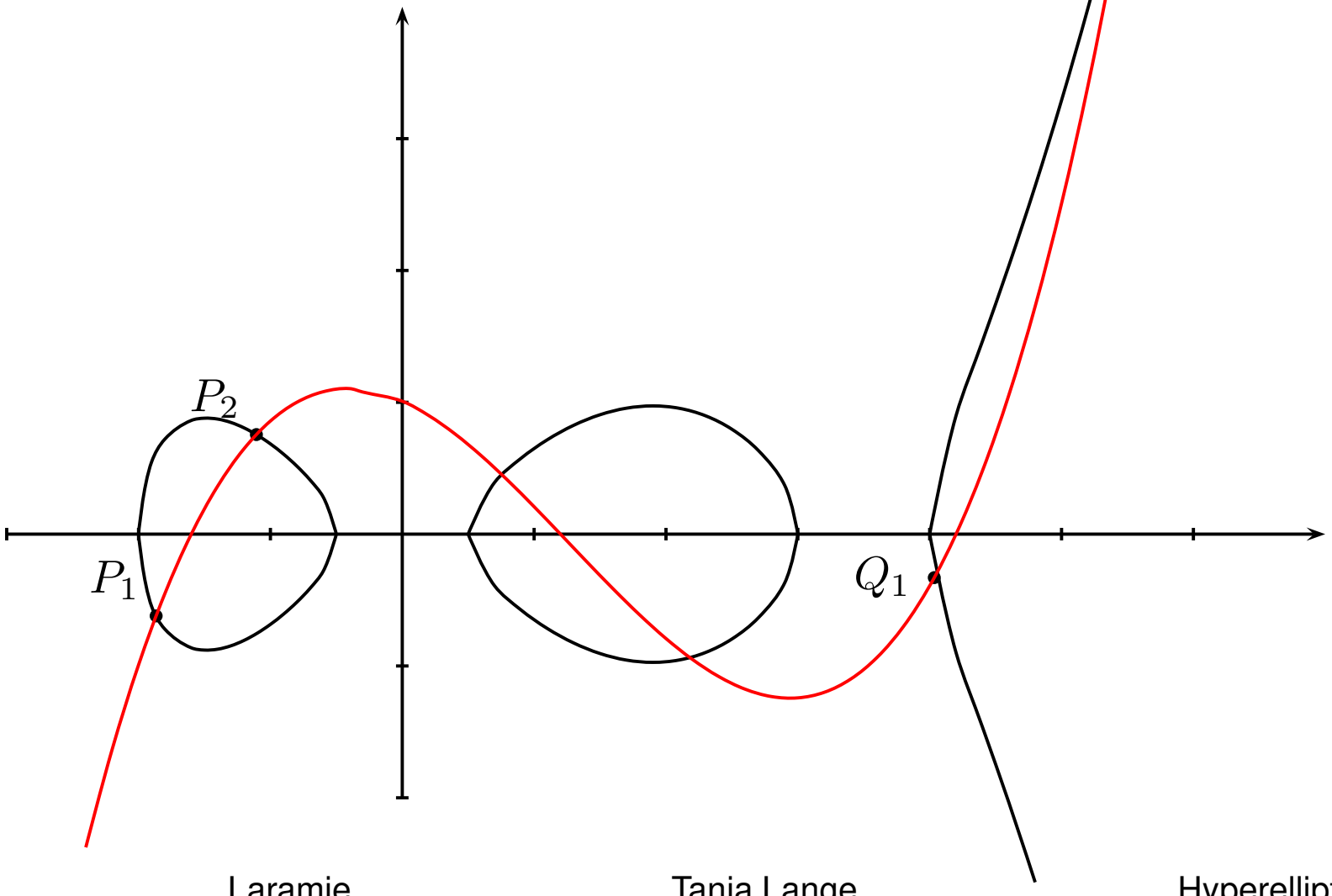
If $\bar{D}$ is defined over $\mathbb{F}_q$, the extension degree of the field of definition of the P_i is bounded, e. g. at most 2 for $g = 2$.

Curve of genus 2 over $\mathbb{R}$, $h = 0$



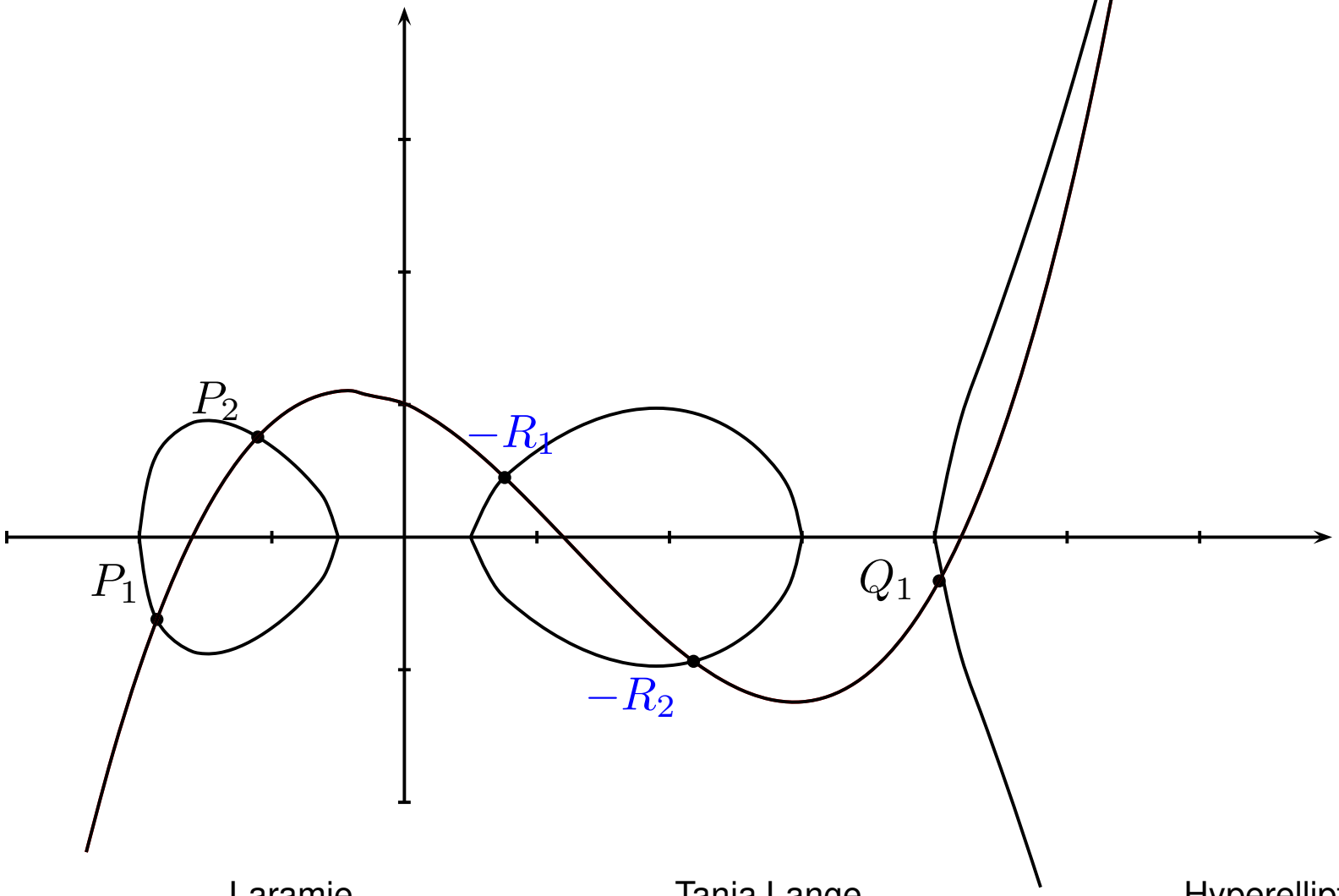
Curve of genus 2 over $\mathbb{R}$, $h = 0$

points on red line $(-6P_\infty)$ form principal divisor



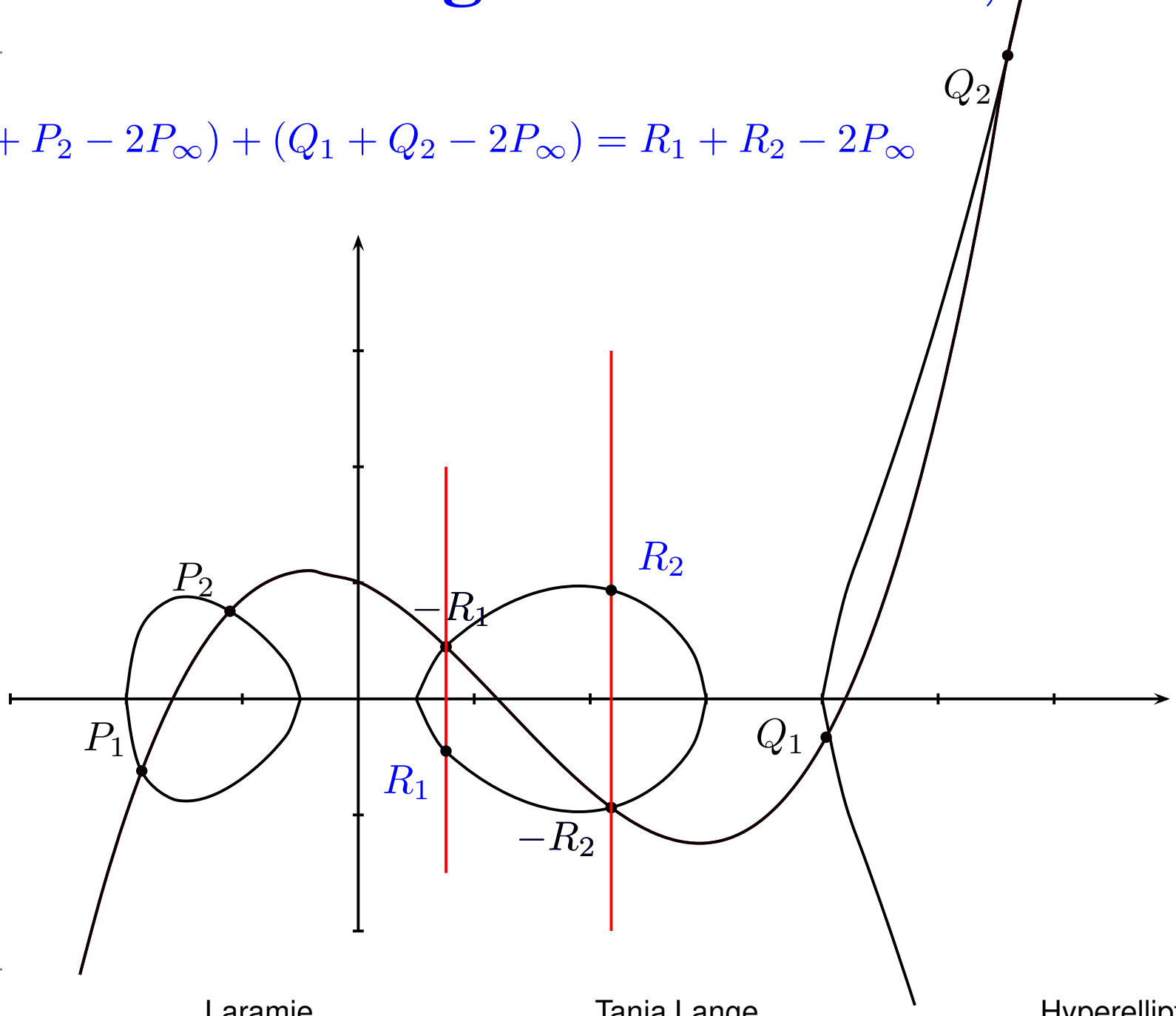
Curve of genus 2 over $\mathbb{R}$, $h = 0$

$P_1 + P_2 + (-R_1) + (-R_2) + Q_1 + Q_2 - 6P_\infty = 0$



Curve of genus 2 over $\mathbb{R}$, $h = 0$

$$(P_1 + P_2 - 2P_\infty) + (Q_1 + Q_2 - 2P_\infty) = R_1 + R_2 - 2P_\infty$$



Ideal Class Group I

- Let $\mathcal{O} = \mathbb{F}_q[x,y]/\left(y^2 + h(x)y - f(x)\right)$.
- Like $\mathbb{Z}$ the ring $\mathbb{F}_q[x]$ is a principal ideal domain (PID), $\mathcal{O}$ is a Dedekind ring (ideal has at most 2 generators).
- Prime ideals in $\mathbb{F}_q[x]$ are generated by irreducible polynomials $u(x)$.
- If (u) splits in $\mathcal{O}$ then there is a polynomial $v(x)$ so that $\langle u(x), y - v(x)\rangle$ and $\langle u(x), y + v(x) + h(x)\rangle$ divide $(u(x))$.
- The ideal class group is the group of ideals modulo the principal ideals.

Ideal Class Group II

- In $\overline{\mathbb{F}}_q[x,y]/\left(y^2+h(x)y-f(x)\right)$ each prime ideal is given by $\langle x-a, y-b\rangle$, this prime lies over $(x-a)$. This implies

$$\begin{aligned}
(x-a) \quad &| \quad (y-b)(y-(-b-h(x))) \\
&\Leftrightarrow \quad (y-b)(y-(-b-h(x))) \equiv 0 \bmod (x-a) \\
&\Leftrightarrow \quad y^2+yh(a)-b^2-h(x)b \equiv 0 \bmod (x-a) \\
&\Leftrightarrow \quad f(x)-b^2-h(x)b \equiv 0 \bmod (x-a) \\
&\Leftrightarrow \quad f(a)-b^2-h(a)b = 0,
\end{aligned}$$

  i.e. $(a,b) \in C(\overline{\mathbb{F}}_q)$.

- Get decomposition of ideals into products of prime ideals – correspondence of divisors and ideals.
- This correspondence obviously only covers affine points. If only one P_∞ then no loss of information.

Mumford Representation

Divisor class group is isomorphic to ideal class group $\mathbf{Cl}(C/\mathbb{F}_q)$ of $\mathbb{F}_q(x,y)/(C)$.
Elements D represented by two polynomials

$$D = [u(x), v(x)]; u, v \in \mathbb{F}_q[x],$$

u monic, $\deg v < \deg u \leq g, \quad u | v^2 + vh - f.$

$\Rightarrow$ compact representation.

Isomorphism is explicit via

$$P_i = (x_i, y_i) \Leftrightarrow u(x_i) = 0, v(x_i) = y_i \text{ with multiplicity}$$

Arithmetic uses *"only"* arithmetic on polynomials . . . but is far less efficient than on elliptic curves (if applied directly).

Arithmetic on hyperelliptic curves

Composition & Reduction (Cantor/Koblitz)

IN: $D_1 = [u_1, v_1], D_2 = [u_2, v_2], C : y^2 + h(x)y = f(x)$
OUT: $D = [u, v]$ reduced with $D \sim D_1 + D_2$

1. compute $d_1 = \gcd(u_1, u_2) = e_1 u_1 + e_2 u_2$
2. compute $d = \gcd(d_1, v_1 + v_2 + h) = c_1 d_1 + c_2(v_1 + v_2 + h)$
3. let $s_1 = c_1 e_1, s_2 = c_1 e_2, s_3 = c_2$
4. $u = \frac{u_1 u_2}{d^2} \quad v = \frac{s_1 u_1 v_2 + s_2 u_2 v_1 + s_3(v_1 v_2 + f)}{d} \bmod u$
5. let $u' = \frac{f - vh - v^2}{u} \quad b' = (-h - v) \bmod u'$
6. if $\deg u' > g$ put $u := u', v := v'$ goto step 5
7. make u monic

Explicit Arithmetic in $\mathbf{Cl}(C/\mathbb{F}_q)$

2000 Harley (odd char.)
2001 L. (arbitrary char.)
2001 Matsuo, Chao, Tsujii (faster)
$\Rightarrow$ 2 inversions

2002 Miyamoto, Doi, Matsuo, Chao, Tsujii
2002 Takahashi
2002 L. (arbitrary char.)
2002 Sugizaki, Matsuo, Chao, Tsujii (even char.)
$\Rightarrow$ 1 inv.

General formulae lead to speeds comparable to ECC (Lange, Avanzi).

2004 L., Stevens $\Rightarrow$ distinction of 3 different cases, optimal affine doubling formulae for each.

General doubling, $g = 2$

Doubling, $\deg u = 2$	
Input	$[u, v], u = x^2 + u_1 x + u_0, v = v_1 x + v_0$
Output	$[u', v'] = 2[u, v]$

Step	Expression	odd	even
1	compute $\tilde{v} \equiv (h + 2v) \bmod u = \tilde{v}_1 x + \tilde{v}_0$: $\tilde{v}_1 = h_1 + 2v_1 - h_2 u_1$, $\tilde{v}_0 = h_0 + 2v_0 - h_2 u_0$;		
2	compute resultant $r =$res$(\tilde{v}, u)$: $w_0 = v_1^2$, $w_1 = u_1^2$, $w_2 = \tilde{v}_1^2$, $w_3 = u_1 \tilde{v}_1$, $r = u_0 w_2 + \tilde{v}_0(\tilde{v}_0 - w_3)$;	2S, 3M $(w_2 = 4w_0)$	2S, 3M (see below)
3	compute almost inverse $inv' = invr$: $inv'_1 = -\tilde{v}_1$, $inv'_0 = \tilde{v}_0 - w_3$;		
4	compute $k' = (f - hv - v^2)/u \bmod u = k'_1 x + k'_0$: $w_3 = f_3 + w_1$, $w_4 = 2u_0$, $k'_1 = 2(w_1 - f_4 u_1) + w_3 - w_4 - h_2 v_1$; $k'_0 = u_1(2w_4 - w_3 + f_4 u_1 + h_2 v_1) + f_2 - w_0 - 2f_4 u_0 - h_1 v_1 - h_2 v_0$;	1M	2M (see below)
5	compute $s' = k' inv' \bmod u$: $w_0 = k'_0 inv'_0$, $w_1 = k'_1 inv'_1$, $s'_1 = (inv'_0 + inv'_1)(k'_0 + k'_1) - w_0 - w_1(1 + u_1)$, $s'_0 = w_0 - u_0 w_1$;	5M	5M
6	compute $s'' = x + s_0/s_1$ and s_1: $w_1 = 1/(rs'_1)(= 1/r^2 s_1)$, $w_2 = rw_1(= 1/s'_1)$, $w_3 = s'^2_1 w_1(= s_1)$; $w_4 = rw_2(= 1/s_1)$, $w_5 = w_4^2$, $s''_0 = s'_0 w_2$;	I, 2S, 5M	I, 2S, 5M
7	compute $l' = s''u = x^3 + l'_2 x^2 + l'_1 x + l'_0$: $l'_2 = u_1 + s''_0$, $l'_1 = u_1 s''_0 + u_0$, $l'_0 = u_0 s''_0$;	2M	2M
8	compute $u' = s^2 + (h + 2v)s/u + (v^2 + hv - f)/u^2$: $u'_0 = s''^2_0 + w_4(h_2(s''_0 - u_1) + 2v_1 + h_1) + w_5(2u_1 - f_4)$, $u'_1 = 2s''_0 + h_2 w_4 - w_5$;	S, 2M	S, M
9	compute $v' \equiv -h - (l + v) \bmod u' = v'_1 x + v'_0$: $w_1 = l'_2 - u'_1$, $w_2 = u'_1 w_1 + u'_0 - l'_1$, $v'_1 = w_2 w_3 - v_1 - h_1 + h_2 u'_1$; $w_2 = u'_0 w_1 - l'_0$, $v'_0 = w_2 w_3 - v_0 - h_0 + h_2 u'_0$;	4M	4M
total		each I, 5S, 22M	

Different Coordinates $g = 2$, odd q

affine $\mathcal{A}$:

$$D \sim [x^2 + u_1 x + u_0, v_1 x + v_0] \sim [u_1, u_0, v_1, v_0]$$

projective $\mathcal{P}$: Miyamoto, Doi, Matsuo, Chao, and Tsujii; L.; all 2002

$$D \sim [U_1, U_0, V_1, V_0, Z] \sim [U_1/Z, U_0/Z, V_1/Z, V_0/Z]$$

new $\mathcal{N}$: L. 2002, contains also mixed coordinates

$$D \sim [U_1, U_1, V_1, V_0, Z_1, Z_2] \sim [U_1/Z_1^2, U_0/Z_1^2, V_1/(Z_1^3 Z_2), V_0/(Z_1^3 Z_2)]$$

$\mathcal{P}$ and $\mathcal{N}$: **no inversions**

Comparison - odd characteristic

Doubling		**Addition**	
operation	costs	operation	costs
$2\mathcal{N} = \mathcal{P}$	7S, 38M	$\mathcal{N} + \mathcal{N} = \mathcal{P}$	7S, 51M
$2\mathcal{P} = \mathcal{P}$	6S, 38M	$\mathcal{N} + \mathcal{P} = \mathcal{P}$	4S, 51M
$2\mathcal{N} = \mathcal{N}$	7S, 34M	$\mathcal{N} + \mathcal{N} = \mathcal{N}$	7S, 47M
$2\mathcal{P} = \mathcal{N}$	6S, 34M	$\mathcal{N} + \mathcal{P} = \mathcal{N}$	4S, 48M
		$\mathcal{P} + \mathcal{P} = \mathcal{P}$	4S, 47M
		$\mathcal{P} + \mathcal{P} = \mathcal{N}$	4S, 44M
		$\mathcal{A} + \mathcal{N} = \mathcal{P}$	5S, 40M
		$\mathcal{A} + \mathcal{P} = \mathcal{P}$	3S, 40M
		$\mathcal{A} + \mathcal{N} = \mathcal{N}$	5S, 36M
		$\mathcal{A} + \mathcal{P} = \mathcal{N}$	3S, 37M
$2\mathcal{A} = \mathcal{A}$	1I, 5S, 22M	$\mathcal{A} + \mathcal{A} = \mathcal{A}$	1I, 3S, 22M

Montgomery Arithmetic

Montgomery Ladder (Arbitrary Group)

Consider the intermediate results (i is decreasing).
$Q_i = \sum_{j=i}^{l} n_j 2^{j-i} P$, put $R_i = Q_i + P$, then

$$Q_i = 2Q_{i+1} + n_i P = Q_{i+1} + R_{i+1} + n_i P - P = 2R_{i+1} + n_i P - 2P.$$

This implies

$$(Q_i, R_i) = \begin{cases} (2Q_{i+1}, Q_{i+1} + R_{i+1}) & \text{if } n_i = 0 \\ (Q_{i+1} + R_{i+1}, 2R_{i+1}) & \text{if } n_i = 1 \end{cases}$$

$$13 = (1101)_2 \sim \begin{array}{rcl} (Q_3, R_3) & = & (P, 2P) \\ (Q_2, R_2) & = & (3P, 4P) \\ (Q_1, R_1) & = & (6P, 7P) \\ (Q_0, R_0) & = & (13P, 14P) \end{array}$$

Montgomery Arithmetic for EC

Advantages:

- Needs only x coordinate (not y)
  $\Rightarrow$ lower storage requirement compared to full Montgomery ladder.
- Projective version (no inversion) extremely efficient for special curves (uses x and z) $\Rightarrow$ Montgomery curves (curves were proposed for ECM factoring due to fast group operation – cf. talk by Dan Bernstein).
- Shielded against simple side-channel attacks (Brier/Joye for arbritrary curves),
  slower than for general curves but faster than full doubling and addition and less storage needed.

Montgomery Arithmetic for Genus 2 Curve

Pierrick Gaudry, ePrint Report 2005/314
Fast genus 2 arithmetic based on Theta functions

- builds on Chudnovsky/Chudnovsky
- uses theta functions to map on Kummer surface, hence different description from Duquesne
- needs full 2-torsion group, i. e. cofactor 16
- shows that approach valid over general fields
- ADD + DBL = 25M – no inversion!!!
  (cf. affine ADD: 22M + 3S + 1I, DBL: 22M + 5S + 1I)
- is faster than elliptic curves in Montgomery form

End of TV part