Code-based cryptography

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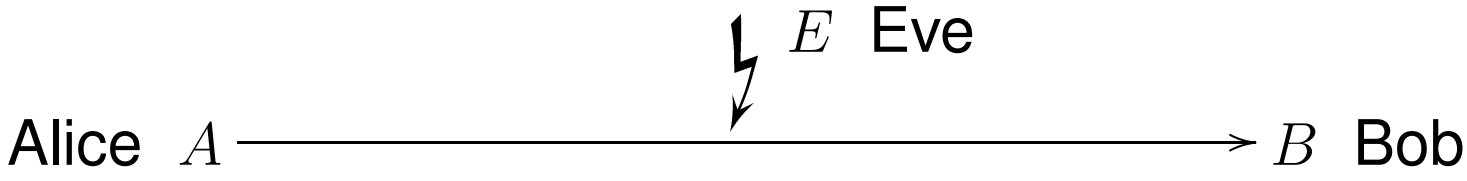
joint work with Daniel J. Bernstein and Christiane Peters

Data Protection

Many factors influence the security and privacy of data

- Secure storage, physical security.
- Access control.
- Protection against alteration of data
  $\Rightarrow$ digital signatures, message authentication codes.
- Protection of sensitive content against reading
  $\Rightarrow$ encryption.



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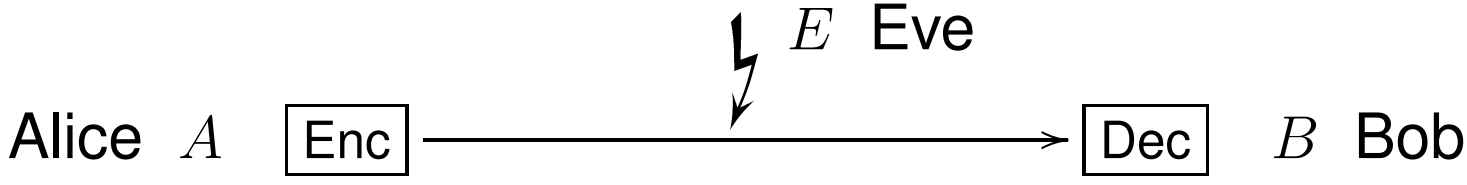

Encryption

Important Distinction

- Partners share a secret (key)
  $\Rightarrow$ use of *symmetric encryption* methods
  - classical methods (Caesar, . . .) belong to this class
  - One-time pad
  - AES
- Partners have no initial secret
  $\Rightarrow$ use of public key cryptography
  - RSA
  - Diffie-Hellman key-exchange
  - ECDSA (elliptic curve digital signature algorithm)

Most applications are hybrid, i.e., they use a combination of both (e.g. use public key cryptography to get session key).

Public Key cryptography

Each user has two keys

- a private key chosen by him which *never* leaves the secure environment,
- a public key which is made public in a kind of phone book.
- These keys are linked by a (computational) one-way function; the general principles are public.
- Sender uses public key of recipient to encrypt
  $\Rightarrow$ only recipient knows secret key and can decrypt.
- Sender uses his private key to sign a message
  $\Rightarrow$ signature must be due to him as only he knows the secret; every recipient can verify the signature using the sender's public key.

Threat of quantum computers

Shor's algorithm makes polynomial time:

- integer factorization
- DLP in finite fields
- DLP on elliptic curves
- DLP in general class groups

Grover's algorithm brings faster simultaneous search in data

- some security loss in symmetric crypto (block and stream ciphers)
- some security loss in hash functions (if not VSH)

Compensate for Grover by $\approx$ doubling key size.

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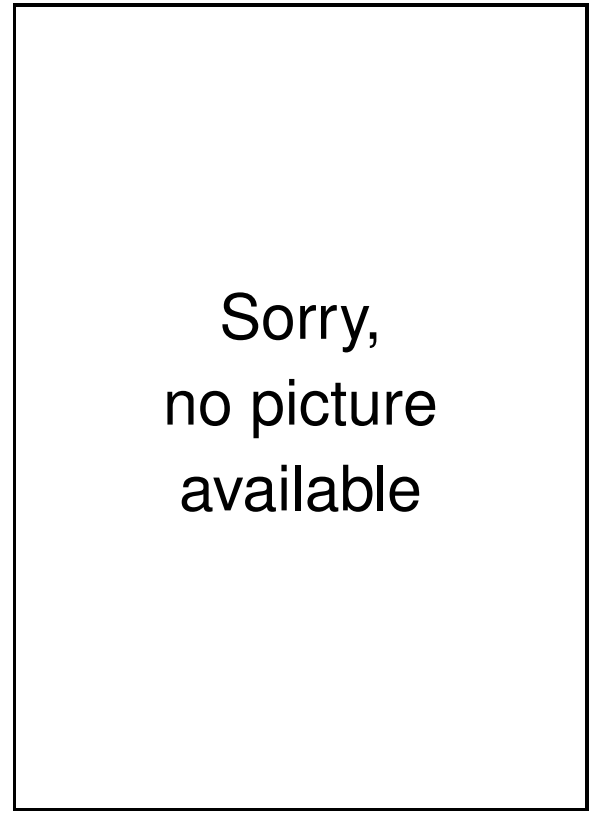

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. . . but 15 years from now . . .

Large quantum computers might be reality. Then

- RSA is dead.
- DH key exchange is dead.
- DSA is dead.
- XTR is dead.
- ECDSA is dead.
- ECC is dead.
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- all public key cryptography is dead?
- Actually there are a few more public-key cryptosystems.

The "survivors"

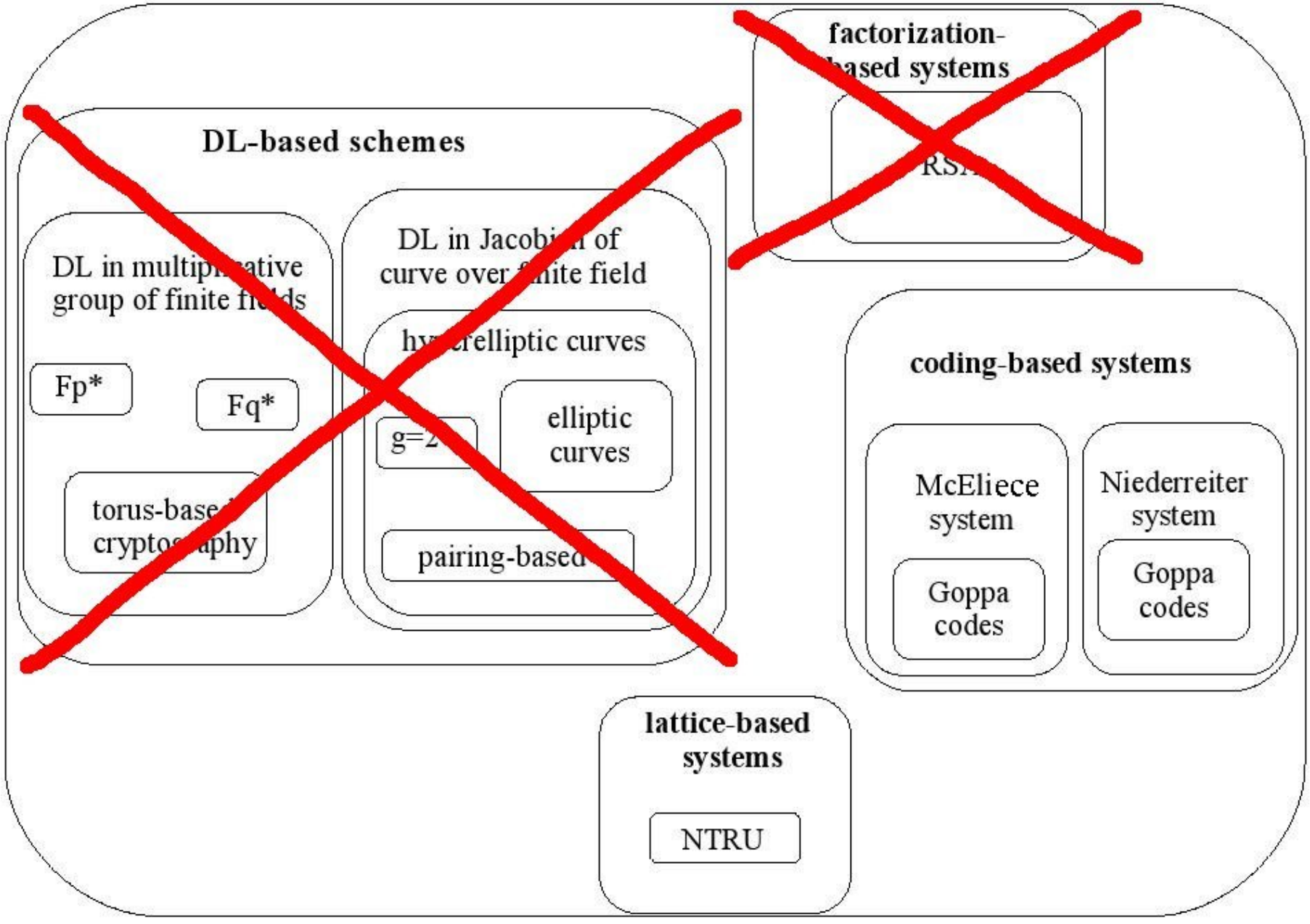
Public-key encryption:

- Lattice-based cryptography (e.g. NTRU)
- Code-based cryptography (e.g. McEliece, Niederreiter)

Public-key signatures:

- Multivariate-quadratic-equations cryptography (e.g. HFE^-)
- Hash based cryptography (e.g. Merkle's hash-trees signatures)

For these systems no efficient usage of Shor's algorithm is known. Grover's algorithm has to be taken into account when choosing key sizes.
Some more possibilities with less confidence.



Encryption systems.

Why care about this now?

15 years might seem a long time. But

- There is no guarantee that it takes at least 15 years.
- Long-term confidential documents (e.g. health records, state secrets) become readable once quantum computers are available.
- Electronic signatures on long-term commitments (e.g. last wishes, contracts) can be forged once quantum computers are available.
- Nobody will inform you if a secret agency made a breakthrough in constructing a quantum computer.
- The systems mentioned before remain secure – but need better study of parameters, faster implementations, and better embedding into protocols.

Post-quantum cryptography

- Cryptographic systems that run on conventional computers, are secure against attacks with conventional computers, and remain secure under attacks with quantum computers are called post-quantum cryptosystems.
- Post-quantum cryptography deals with
  - the design of such systems;
  - cryptanalysis of such systems;
  - the analysis of suitable parameters depending on different threat models;
  - design of protocols using the secure primitives.
- Could potentially develop cryptosystems for quantum computers – but there won't be a market too soon.

Warnings

- The following describes text-book versions.
- There exist CCA2 secure versions, this does not change the mathematics behind it or the mathematical ways of attacking the systems.

Linear codes – notation

- A generator matrix of an $[n, k]$ code C is a $k \times n$ matrix G such that $C = \{\mathbf{x}\,G : \mathbf{x} \in \mathbb{F}_q^k\}$.
- The matrix G corresponds to a map $\mathbb{F}_q^k \to \mathbb{F}_q^n$ sending a message of length k to an n-bit string.
- A parity-check matrix of an $[n, k]$ code C is an $(n-k) \times n$ matrix H such that $C = \{\mathbf{c} \in \mathbb{F}_q^n : H\,\mathbf{c}^T = 0\}$.
- A systematic generator matrix is a generator matrix of the form $(I_k|Q)$ where I_k is the $k \times k$ identity matrix and Q is a $k \times (n-k)$ matrix (redundant part).
- Easy to get parity-check matrix from systematic generator matrix, use $H = (-Q^T|I_{n-k})$.
- The minimum distance of a linear code C is the smallest Hamming weight of a nonzero codeword in C.

The McEliece cryptosystem

First part focuses on generic attacks, so no special properties assumed about the code.

- Let C be an $[n, k]$ code with minimum distance $2w + 1$ given by a generator matrix G. The McEliece public key is the $k \times n$ matrix G and w.
- The McEliece secret key consists of an efficient w-error correcting decoding algorithm for C.
- McEliece encryption: of a message $\mathbf{m}$ of length k: Compute $\mathbf{m}G$ and add a random error vector $\mathbf{e}$ of weight w and length n. Send $\mathbf{y} = \mathbf{m}G + \mathbf{e}$.

Attacks on the McEliece PKC

- Encryption of message m is $\mathbf{y} = \mathbf{m}G + \mathbf{e}$ with random $\mathbf{e}$ of weight w.
- Attacker is faced with decoding $\mathbf{y}$ to nearest codeword $\mathbf{m}G$. This is decoding a generic code.
- Most effective attack against the McEliece cryptosystem (with binary Goppa code) is information-set decoding.
- Many variants:
  McEliece (1978), Leon (1988), Lee and Brickell (1988), Stern (1989), van Tilburg (1990), Canteaut and Chabanne (1994), Canteaut and Chabaud (1998), Canteaut and Sendrier (1998), Bernstein, Lange, and Peters (2008), and Finiasz and Sendrier (2009).
- New result: Ball-collision decoding with proper analysis and exponential improvement over previous results.

Search for low weight words vs. decoding

- McEliece ciphertext $\mathbf{y} \in \mathbb{F}_q^n$ has distance w from a unique closest codeword $\mathbf{c} = \mathbf{m}G$ in a code C which has minimum distance at least $2w + 1$.
- Find $\mathbf{e}$ of weight w such that $\mathbf{c} = \mathbf{y} - \mathbf{e}$:
  - append $\mathbf{y}$ to the list of generators
  - and form a generator matrix for $C + \{0, \mathbf{y}\}$.

  Then

$$\mathbf{e} = (\mathbf{m}, 1) \begin{pmatrix} G \\ \hline \mathbf{m}G + \mathbf{e} \end{pmatrix}$$

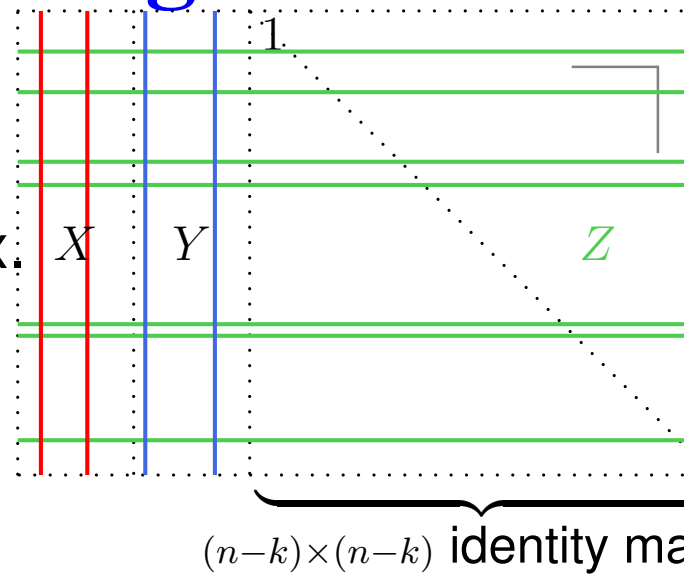
  is a codeword in $C + \{0, \mathbf{y}\}$; and it is the only weight-w word.
- Bottleneck in all of these attacks is finding the weight-w codeword in $C + \{0, \mathbf{y}\}$. This code has dimension $k + 1$.

Stern's attack

- Focus on binary codes, i.e. codes over $\mathbb{F}_2$, here. Generalization (and much more) done by Peters (2010).
- Given $w \geq 0$ and an $(n-k) \times n$ parity check matrix H for a binary $[n,k]$ code C. Goal is to find $\mathbf{c} \in C$ of weight w.
- Construct $\mathbf{c}$ by looking for exactly w columns of H which add up to 0.
  - Select $n-k$ columns and transfer these into identity matrix.
  - Choose two disjoint subsets X, Y among the other columns of H and a subset Z among the first.
  - Search for words having exactly $p, p, 0$ ones in those column sets X, Y, Z and exactly $w-2p$ ones in the remaining columns.
  - Iterate until such a word is found for some X, Y, Z.

One iteration of Stern's algorithm

- Select $n-k$ linearly independent columns; apply elementary row operations to get the identity matrix.
- Form a set Z of ℓ rows (green).
- Divide remaining k columns into two subsets X and Y.



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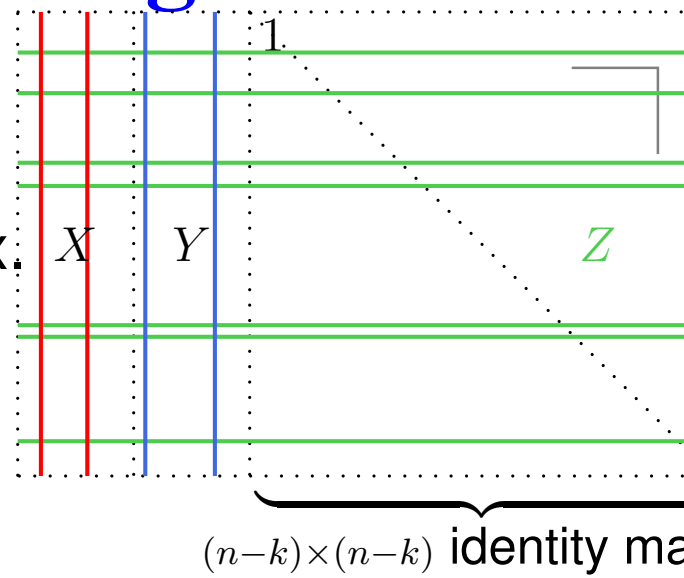

- Select $n-k$ linearly independent columns; apply elementary row operations to get the identity matrix.
- Form a set Z of ℓ rows (green).
- Divide remaining k columns into two subsets X and Y.
- For every size-p subset A of X compute the ℓ-bit vector $\pi(A)$ by adding up the columns of $H' = (H_{i,j})_{i \in Z, j \in A}$ (red). Similarly, compute $\pi(B)$ (blue).
- For each collision $\pi(A) = \pi(B)$ compute the sum of the $2p$ columns in $A \cup B$. This sum is an $(n-k)$-bit vector.

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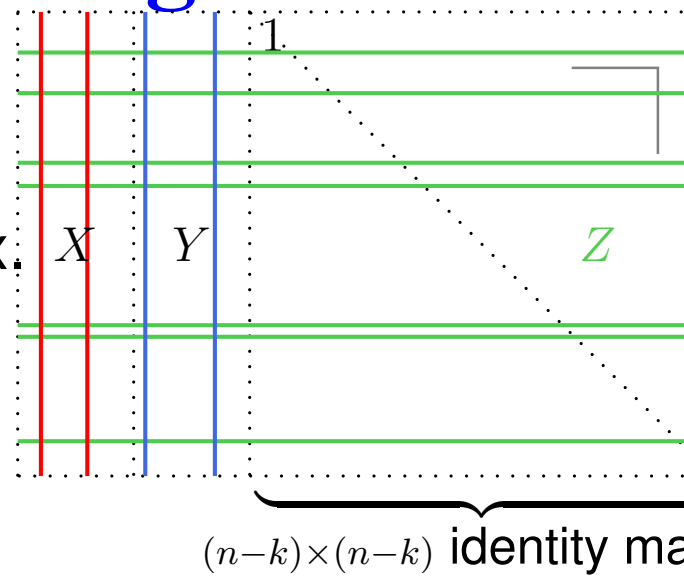

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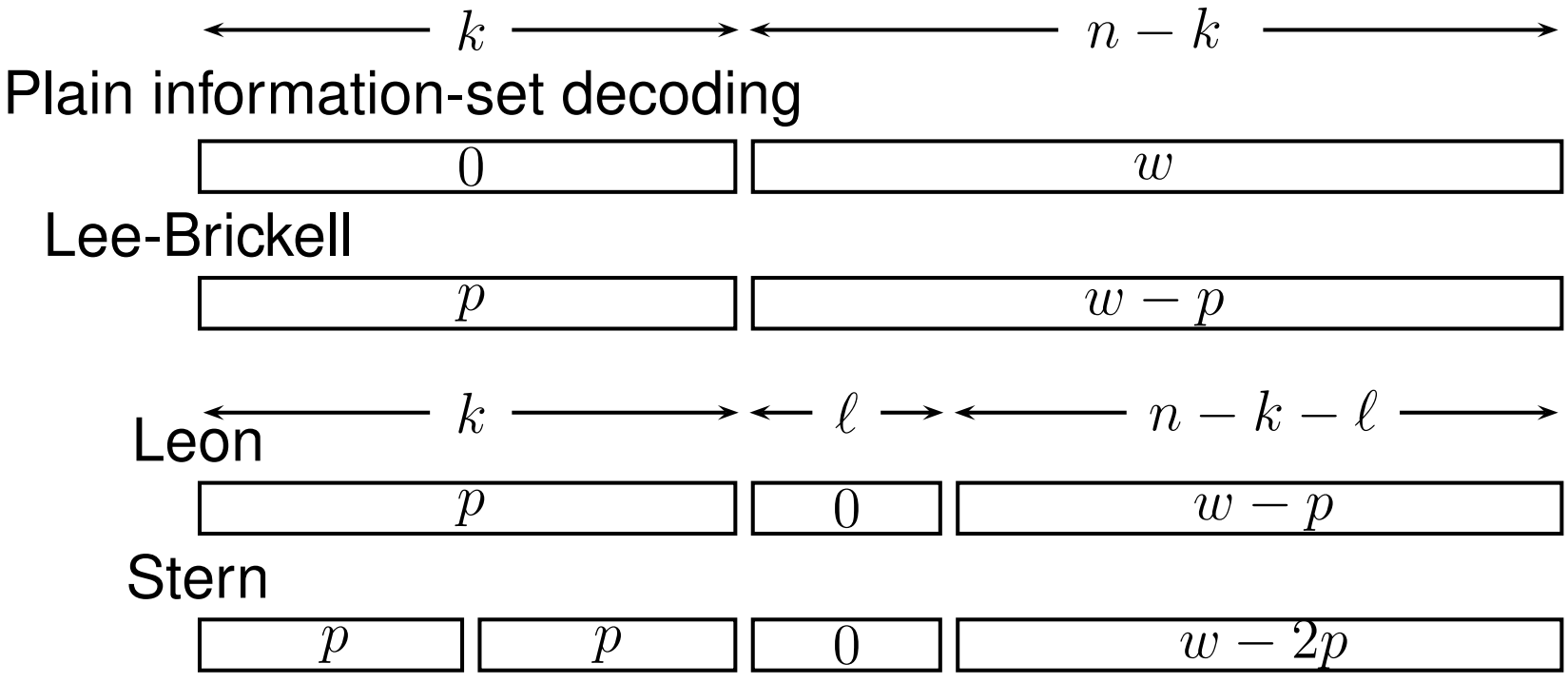
- **If** the sum has weight $w-2p$, we obtain 0 by adding the corresponding $w-2p$ columns in the $(n-k) \times (n-k)$ submatrix. **Else** select $n-k$ new columns.

Running time in practice

- Wrote attack software against original McEliece parameters, decoding 50 errors in a $[1024, 524]$ code.
- Lots of improvements stated in 2008 paper.
- Attack on a single computer with a 2.4GHz Intel Core 2 Quad Q6600 CPU would need, on average, 1400 days (2^{58} CPU cycles) to complete the attack.
- About 200 computers involved, with about 300 cores.
- Most of the cores put in far fewer than 90 days of work; some of which were considerably slower than a Core 2.
- Computation used about 8000 core-days.
- Error vector found by Walton cluster at SFI/HEA Irish Centre of High-End Computing (ICHEC).

Information-set decoding

Methods differ in where the errors are allowed to be.



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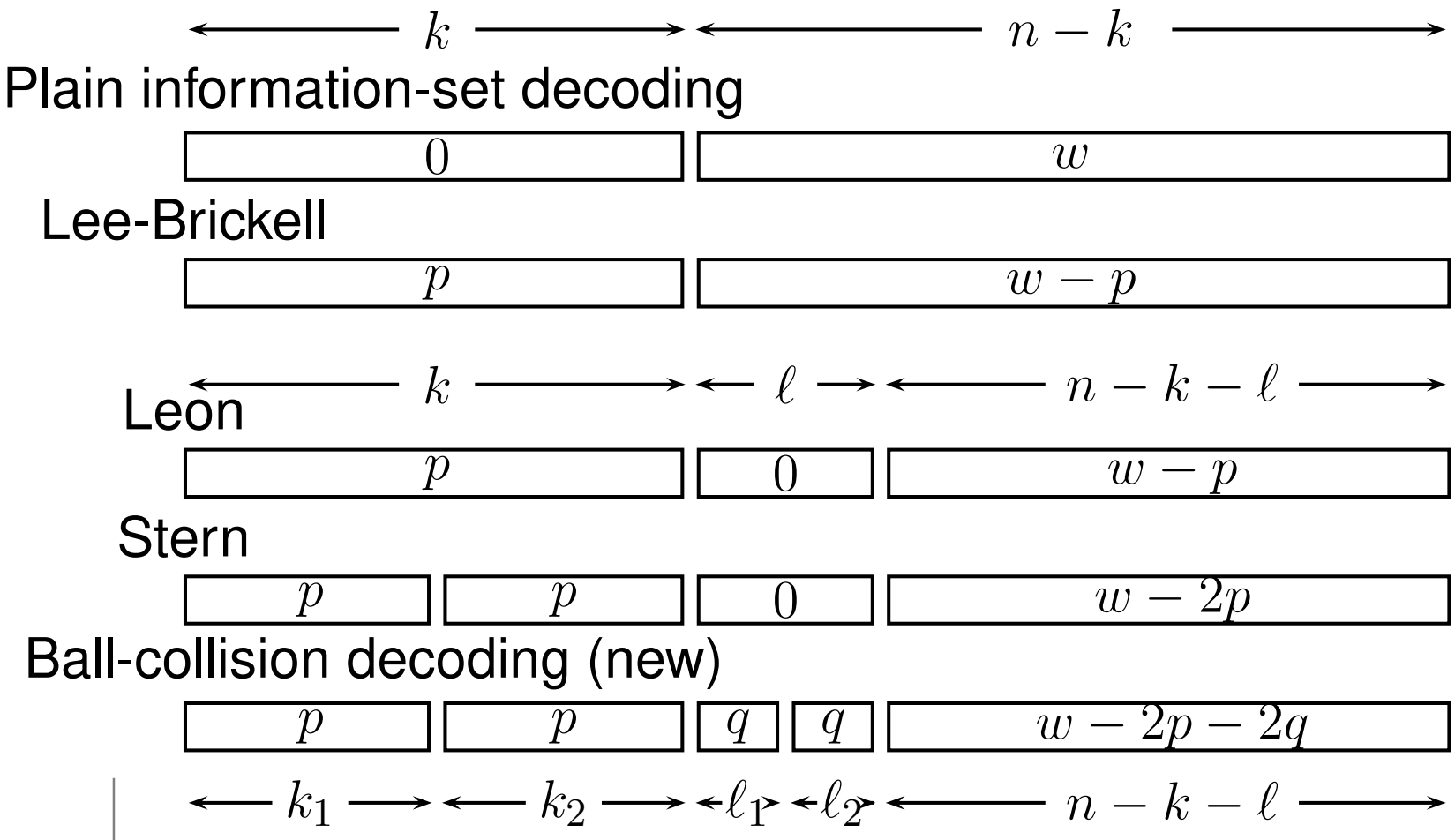

Ball-collision decoding

- Do not insist on having 0 errors in Z but allow $2q$ errors in this stretch.
- This helps! If ℓ is optimized in Stern's algorithm (and relatives) it grows with n, so Stern is successful if an increasingly large stretch of the vector is error free. In hindsight, that's a bad idea.
- This algorithm is inspired by supercode decoding (Barg, Krouk, van Tilborg; 1999). Supercode decoding enforces an increasingly detailed error pattern on the code but the analysis is wrong (in the appendix).
- Decoding of generic codes takes time $2^{(\alpha(R,W)+o(1))n}$, for rate $k/n \to R$ and error-rate $w/n \to W$ for $n \to \infty$.
- Ball-collision decoding achieves a smaller α, i.e. an exponential speedup over Stern's algorithm!

What codes are behind this?

- McEliece suggested to use classical Goppa codes Γ.
- If you know algebraic geometry Goppa codes, these are included as genus-0 codes.
- These codes are efficiently decodeable – so how can there be a secret?
- The McEliece secret key consists of a generator matrix G for Γ, an efficient t-error correcting decoding algorithm for Γ; an $n \times n$ permutation matrix P and a nonsingular $k \times k$ matrix S.
- n, k, t are public; but Γ, P, S are randomly generated secrets. Key size $k(n-k)$ elements in $\mathbb{F}_q$.
- The McEliece public key is the $k \times n$ matrix $G' = SGP$; G' efficiently hides the structure of G.

Goppa codes

- Fix a prime power q; a positive integer m; a positive integer $n \leq q^m$; an integer $t < n/m$;
- fix distinct elements $a_1, \ldots, a_n$ in $\mathbb{F}_{q^m}$;
- fix polynomial $g(x)$ in $\mathbb{F}_{q^m}[x]$, $\deg(g) = t$ s.t. $g(a_i) \neq 0$.
- Denote by $\Gamma_{q^m}(a_1, \ldots, a_n, g)$ the linear code consisting of all words $c = (c_1, \ldots, c_n)$ in $\mathbb{F}_{q^m}^n$ satisfying

$$\sum_{i=1}^{n} \frac{c_i}{x - a_i} \equiv 0 \pmod{g(x)}$$

- The Goppa code $\Gamma_q(a_1, \ldots, a_n, g)$ with Goppa polynomial $g(x)$ and support $a_1, \ldots, a_n$ is the restriction of $\Gamma_{q^m}(a_1, \ldots, a_n, g)$ to $\mathbb{F}_q$.
- Has dimension $\geq n - mt$ and minimum distance $\geq t + 1$.

Dependence on q

- Choose n, m, t so that decoding in G' is hard; larger n and m mean harder to decode codes – but also larger keys.
- For $q = 2$ the minimum distance is at least $2t + 1$. Patterson's algorithm decodes efficiently up to t errors.
- For larger q no such construction known, so can correct only up to $t/2$ errors.
- Generalizations of information-set decoding to $q > 2$ by Peters show that key sizes are larger for $2 < q < 31$ than for $q = 2$ at the same level of security (hardness of decoding G').
- For $q \geq 31$ one can achieve smaller public keys.
- Haven't done implementation yet, so don't have speed comparison for $q = 2$ vs. $q = 31$.

Wild Goppa Codes

- Theorem by Sugiyama, Kasahara, Hirasawa, and Namekawa:
  Let q be a prime power. Let m be a positive integer. Let n be an integer with $1 \leq n \leq q^m$. Let $a_1, a_2, \ldots, a_n$ be distinct elements of $\mathbb{F}_{q^m}$. Let g be a monic squarefree polynomial in $\mathbb{F}_{q^m}[x]$ coprime to $(x - a_1) \cdots (x - a_n)$. Then $\Gamma_q(a_1, a_2, \ldots, a_n, g^{q-1}) = \Gamma_q(a_1, a_2, \ldots, a_n, g^q)$.
- We call these codes wild Goppa codes in analogy to wild ramification in extension fields.
- Put $\deg(g) = t$. Code with polynomial g^{q-1} is the same as code with g^q, so it has minimum distance $\geq qt + 1$ instead of $\geq (q-1)t + 1$.
- This covers well-known result for $q = 2$; gives an important improvement for $q = 3, q = 4, \ldots$

Generalization of wild Goppa Codes

- Even more wild?

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- Potential problem: degree of g (when optimized for key size) decreases with growing q; starting with $q = 11$ searching through all polynomials to find g is faster than information-set decoding for security level 2^{128}.
- Only "potential" problem since support would still be unknown and we can choose that so large that search is harder.
- Have published challenges to see what is easier to break.
- Nevertheless: suggest to use

$$\Gamma_q(a_1, a_2, \ldots, a_n, g^{q-1}f) = \Gamma_q(a_1, a_2, \ldots, a_n, g^q f)$$

- Also offers better granularity in choosing the degrees.

Proof

$\Leftarrow$ If $\sum_i c_i/(x-a_i) \equiv 0 \bmod g^q$ then $\sum_i c_i/(x-a_i) \equiv 0 \bmod g^{q-1}$.

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Use $\frac{1}{x-a_i} = -\frac{1}{a_i-r} - \frac{x-r}{(a_i-r)^2} - \frac{(x-r)^2}{(a_i-r)^3} - \cdots$ to see

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Take the qth power of the equation $\sum_i c_i/(a_i-r) = 0$, and use $c_i \in \mathbb{F}_q$, to obtain $\sum_i c_i/(a_i-r)^q = 0$. Work backwards to see that $\sum_i c_i/(x-a_i) = 0$ in $k[x]/(x-r)^q$.

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All $x-r$ are distinct, thus $\sum_i c_i/(x-a_i) = 0$ in $k[x]/g^q$.

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