Advances in Elliptic-Curve Cryptography

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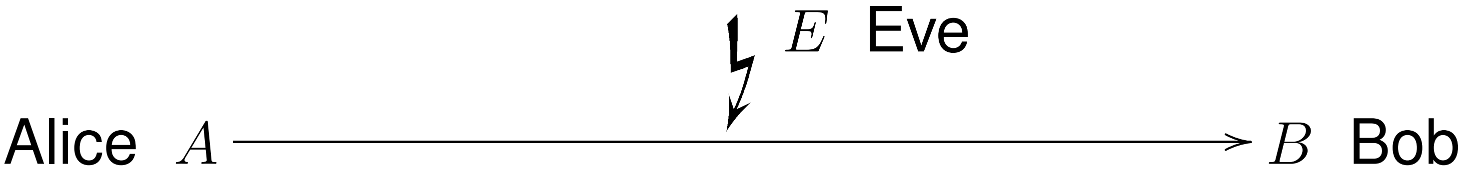
Data Protection

Many factors influence the security and privacy of data

- Secure storage, physical security.
- Access control.
- Protection against alteration of data
  $\Rightarrow$ digital signatures, message authentication codes.
- Protection of sensitive content against reading
  $\Rightarrow$ encryption.



Eve hears everything on the line; she can intercept and modify contents.

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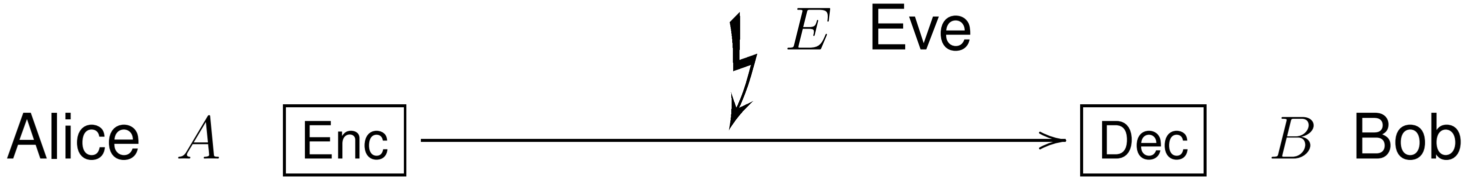

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Public Key cryptography

Each user has two keys

- a private key chosen by him which *never* leaves the secure environment,
- a public key which is made public in a kind of phone book.
- These keys are linked by a (computational) one-way function; the general principles are public.
- Sender uses public key of recipient to encrypt
  $\Rightarrow$ only recipient knows secret key and can decrypt.
- Sender uses her private key to sign a message
  $\Rightarrow$ signature must be due to her as only she knows the secret; every recipient can verify the signature using the sender's public key.

Common set-up

- No initial shared secret; use public key crypto
  - to get session key (e.g. Diffie-Hellman key-exchange);
  - to authenticate a message and ensure its integrity, verifiable by anybody, using digital signatures.
- Symmetric crypto methods with shared key
  - to encrypt the message (e.g.. block cipher AES);
  - to provide authenticity and integrity for the other holder of the key (e.g. message authentication codes);
- Cryptographic hash functions satisfy:
  - hash: $\{0,1\}^* \to \{0,1\}^\ell$ map to fixed length strings;
  - finding preimages and second preimages is hard;
  - finding collisions is hard.

ECDSA I

- Elliptic-Curve Digital Signature Algorithm.
- Suggested by Certicom, standardized by FIPS and others.
- Some other elliptic-curve signature schemes were proposed, e.g. by Korea and by Germany.
- Starting point is an elliptic curve E, in Weierstrass form over a finite field $\mathbb{F}_q$ and a point $P \in E(\mathbb{F}_q)$ of some prime order ℓ.
- Assume for simplicity $q > 3$ is an odd prime,

$$E : y^2 = x^3 + a_4 x + a_6;\ a_4, a_6 \in \mathbb{F}_q$$

and $E(\mathbb{F}_q) = \langle P \rangle$.

ECDSA II

- Alice's secret key is integer $a \in [1, \ell - 1]$ and her public key is $P_A = aP$.
- To sign a message m with hash value $h =$hash(m) she
  - picks random $k \in [1, \ell - 1]$;
  - computes $R = kP$ and puts $r \equiv x(R) \bmod \ell$;
  - computes $s \equiv k^{-1}(h + ra) \bmod \ell$.
  - The signature is the tuple (r, s), both values are smaller than ℓ.
- To verify the signature (r, s) on m Bob computes
  - $w \equiv s^{-1} \bmod \ell, u_1 \equiv hw \bmod \ell, u_2 \equiv rw \bmod \ell$;
  - and $Q = u_1 P + u_2 P_A$;
  - and accepts the signature if $r \equiv x(Q) \bmod \ell$; otherwise he rejects it.

ECDSA III

- Alice's signatures should be accepted, i.e. the x-coordinate of $R = kP$ should be the same as the x-coordinate of $Q = u_1 P + u_2 P_A$, where $P_A = aP$, $w \equiv s^{-1} \bmod \ell, u_1 \equiv hw \bmod \ell, u_2 \equiv rw \bmod \ell$, $r \equiv x(R) \bmod \ell$, and $s \equiv k^{-1}(h + ra) \bmod \ell$.

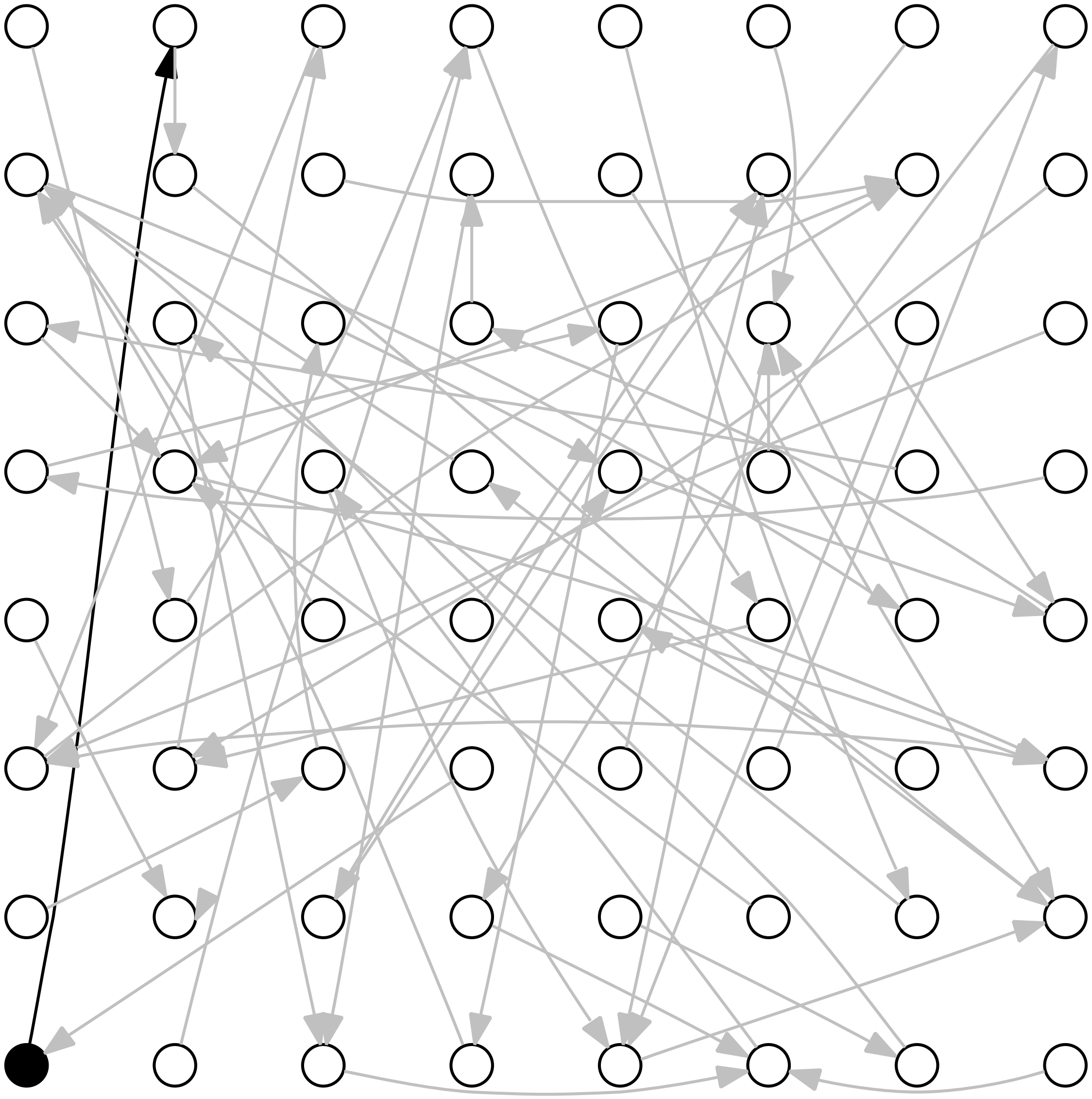
$$\begin{aligned} Q &= u_1 P + u_2 P_A = hwP + rwP_A = s^{-1}(hP + raP) \\ &= (k^{-1}(h + ra))^{-1}(h + ra)P = kP = R \end{aligned}$$

- Note that P having order ℓ takes care of $\mathrm{mod}\ell$ computations. Since $\ell \approx q$, $\pm R$ essentially given by r
- Only secret part in (r, s) is a; attacker Eve can choose any random k.
- Computing a from P and $P_A = aP$ is solving the Discrete Logarithm Problem (DLP) of P_A with base P.

The Pollard rho method

- Standard method to compute a given P and $Q = aP$.
- Expect $(1 + o(1))\sqrt{\pi\ell/2}$ group operations, matching Nechaev/Shoup bound.
- Easy to distribute across CPUs; very little memory consumption; very little communication.
- Simplified, non-parallel rho: Make a pseudo-random walk in $\langle P \rangle$, where the next step depends on current point: $W_{i+1} = f(W_i)$.
- Birthday paradox: Randomly choosing from ℓ elements picks one element twice after about $\sqrt{\pi\ell/2}$ draws.
- The walk now enters a cycle. Cycle-finding algorithm (e.g., Floyd) quickly detects this.

Random walk example



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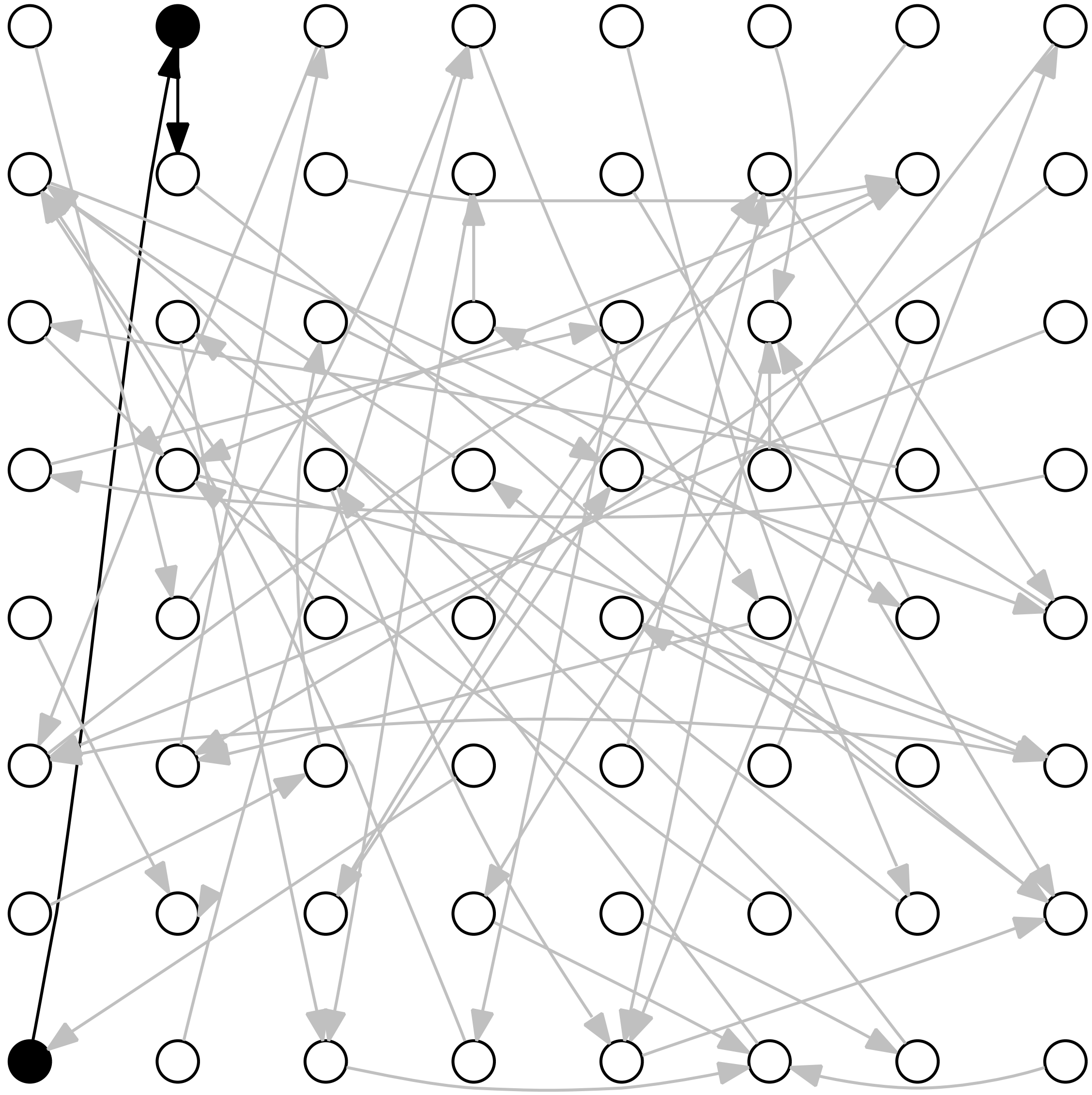

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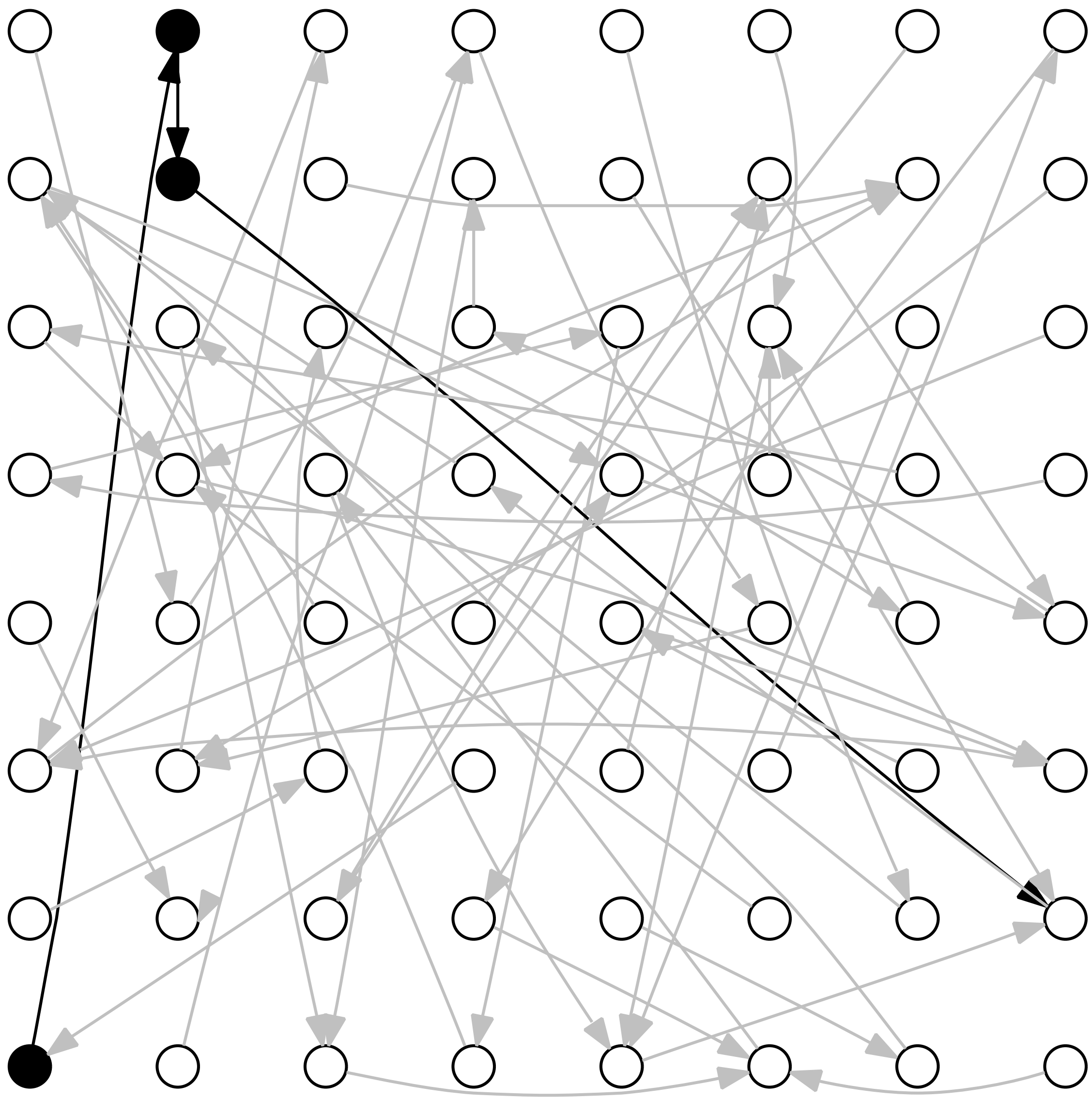

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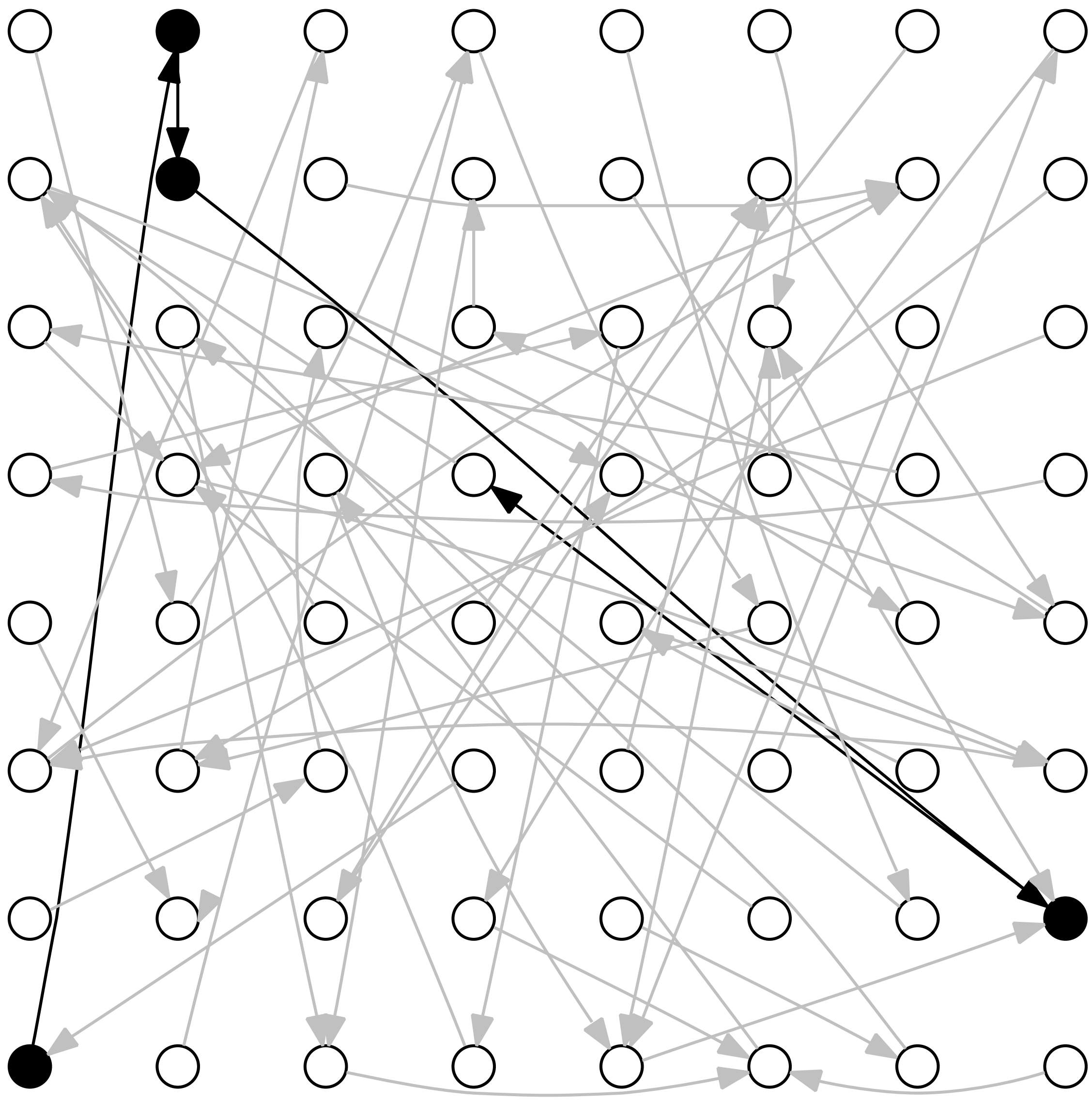

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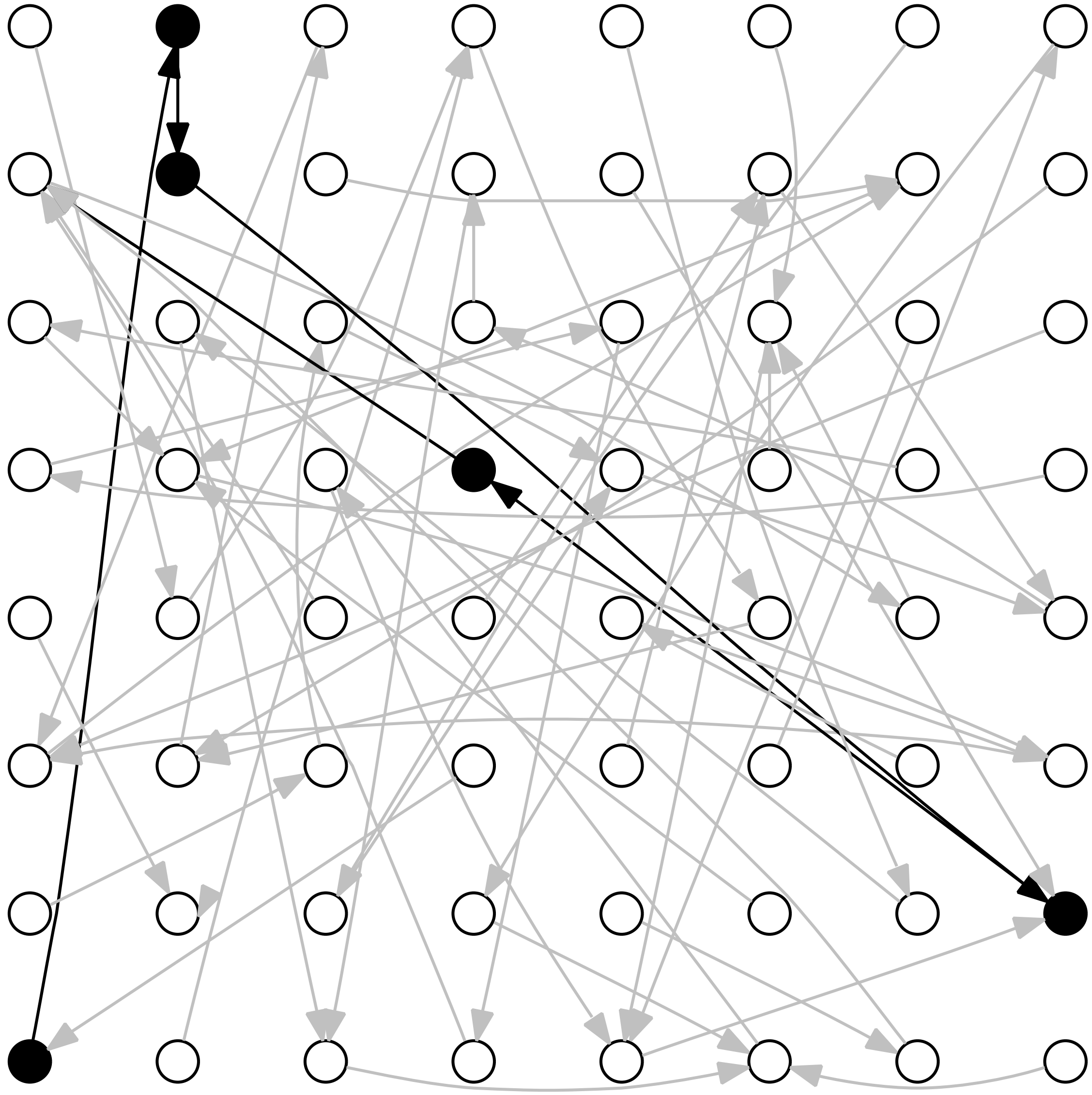

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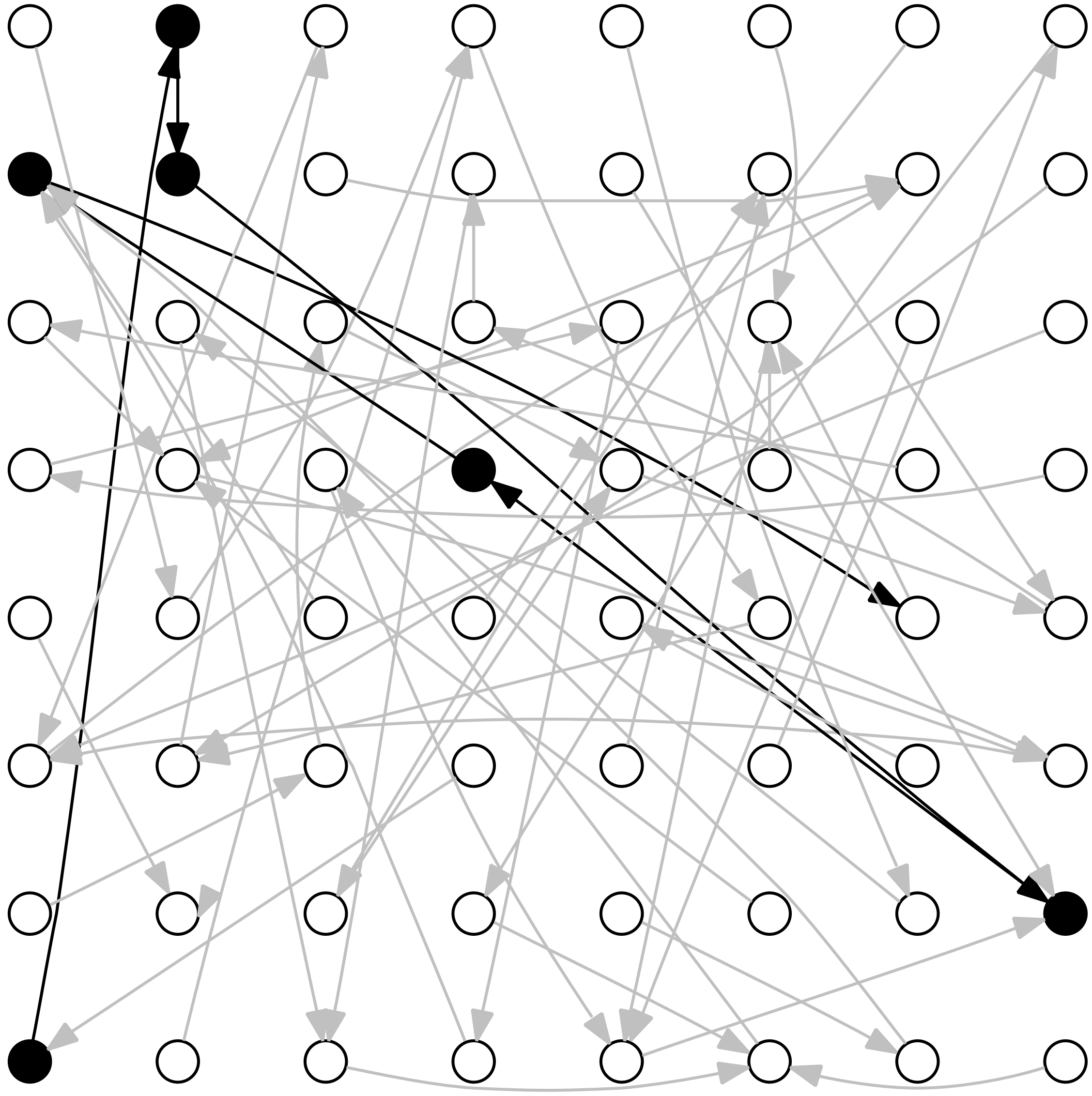

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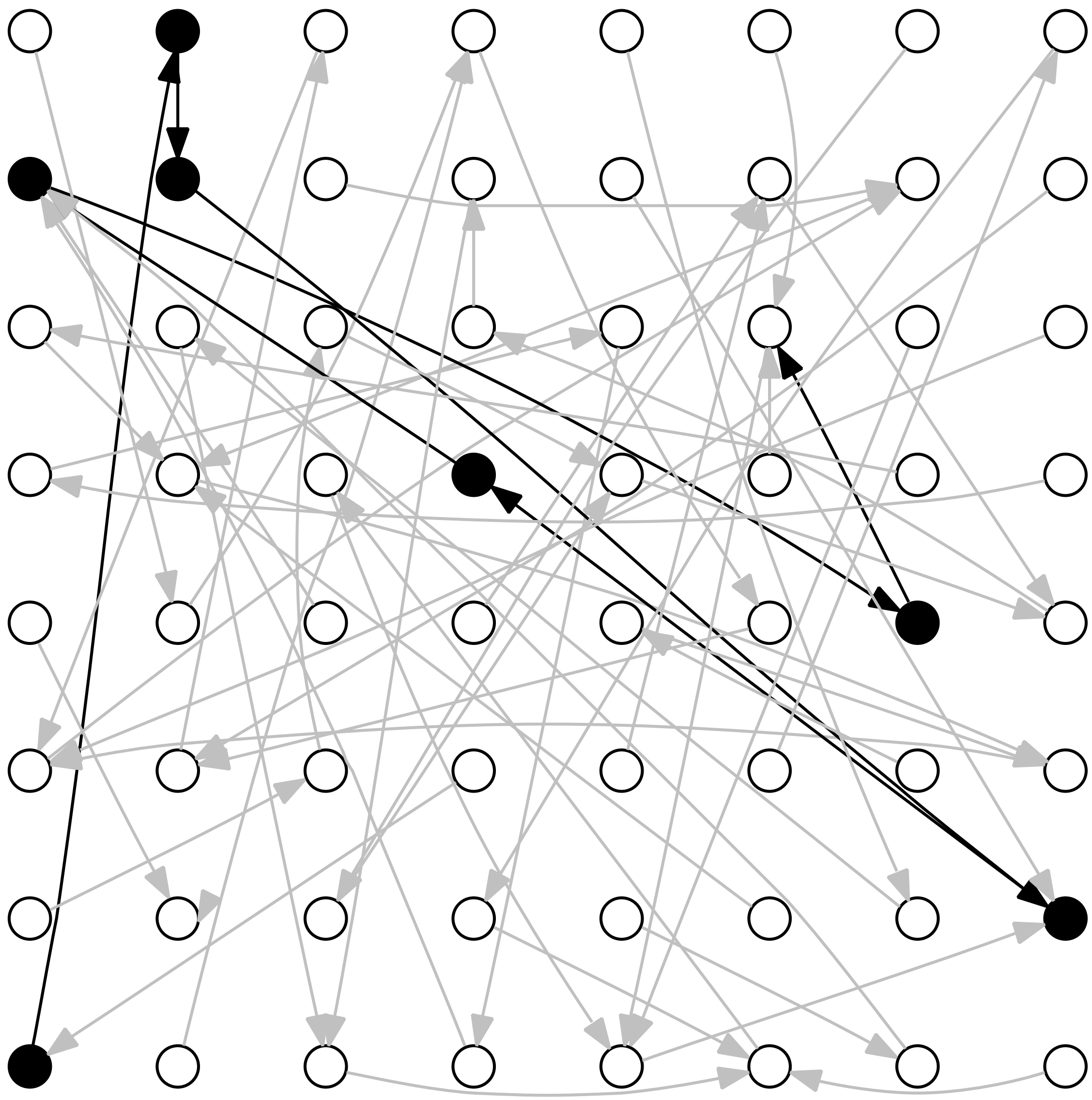

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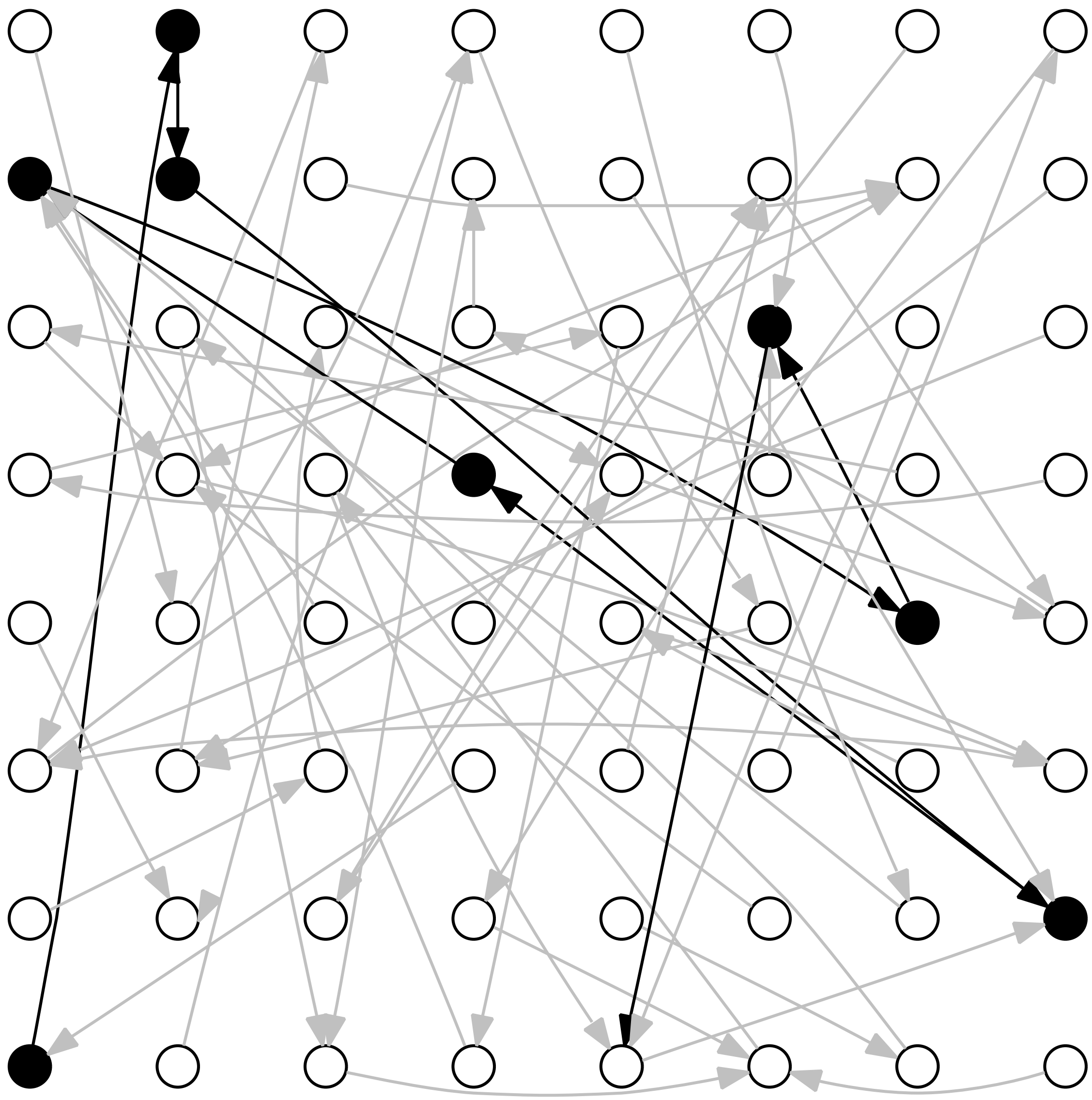

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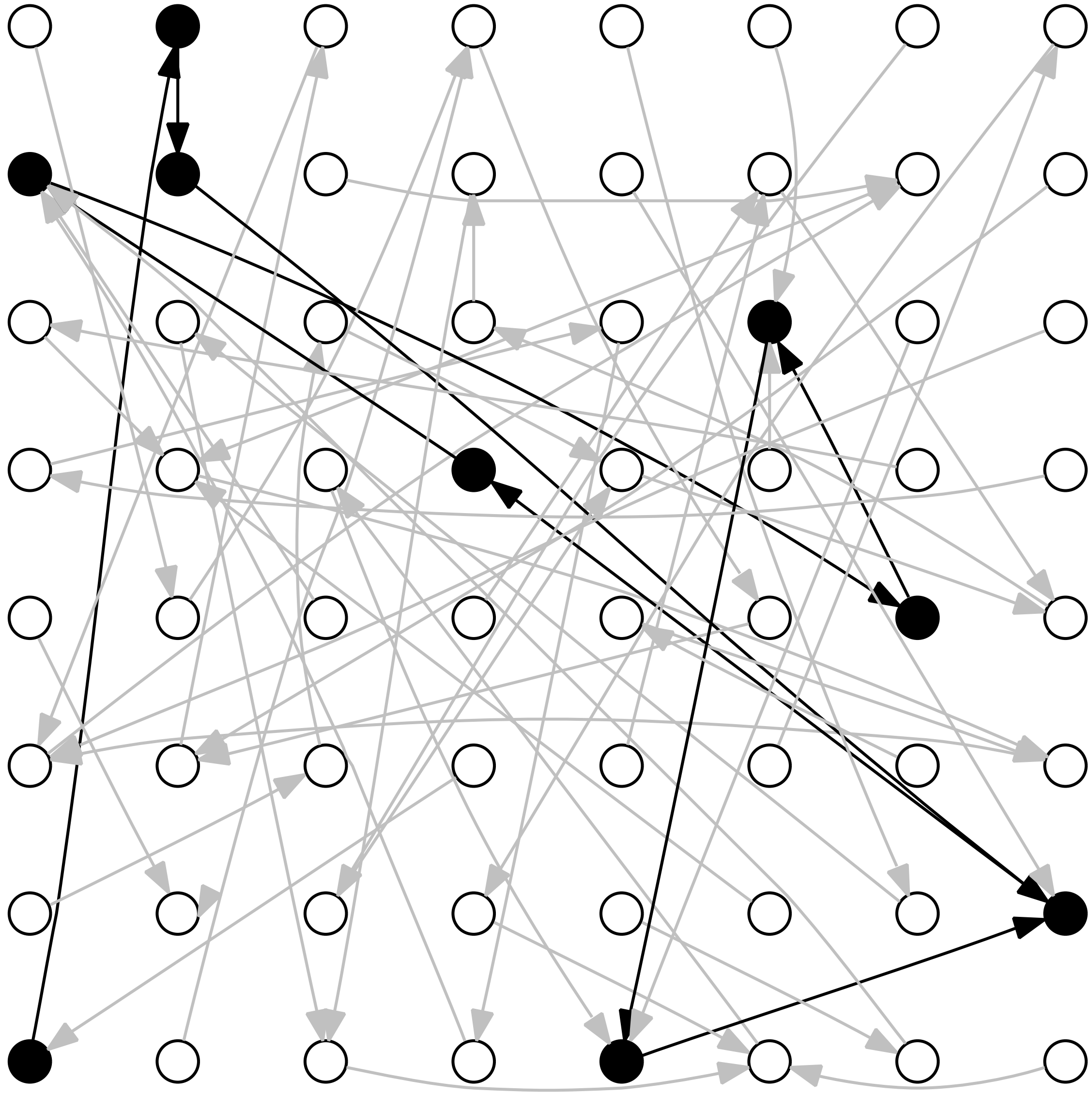

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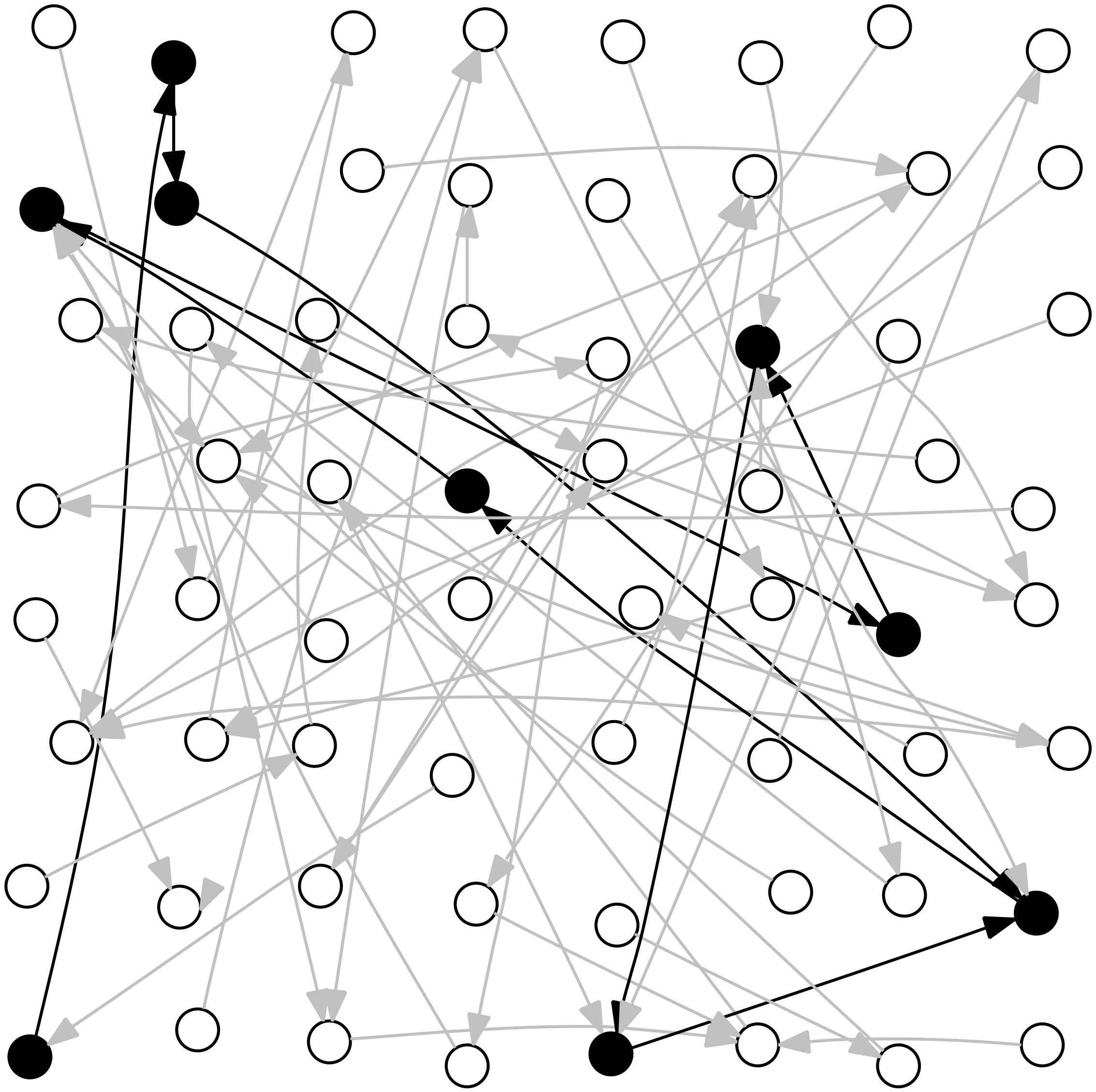

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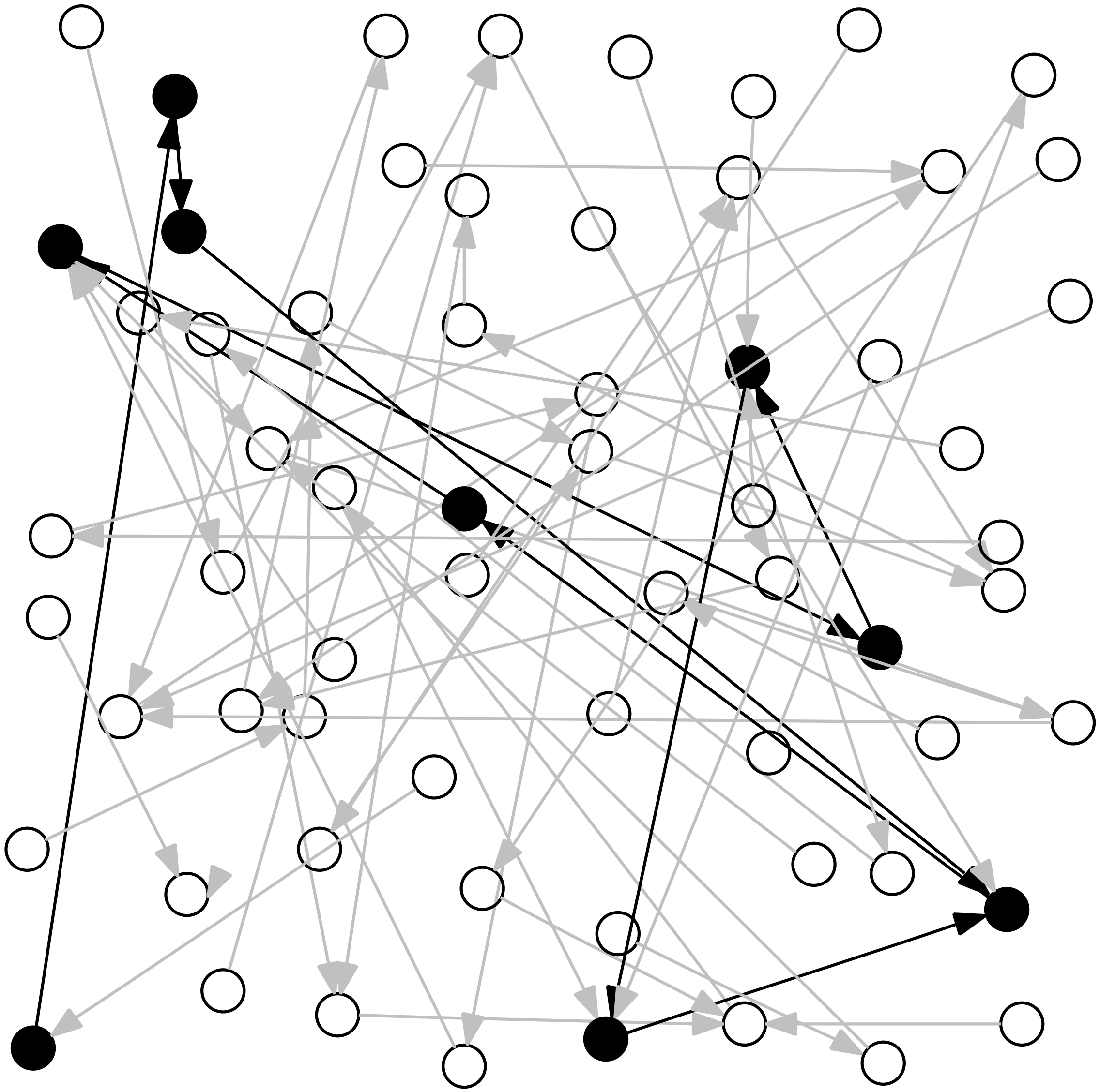

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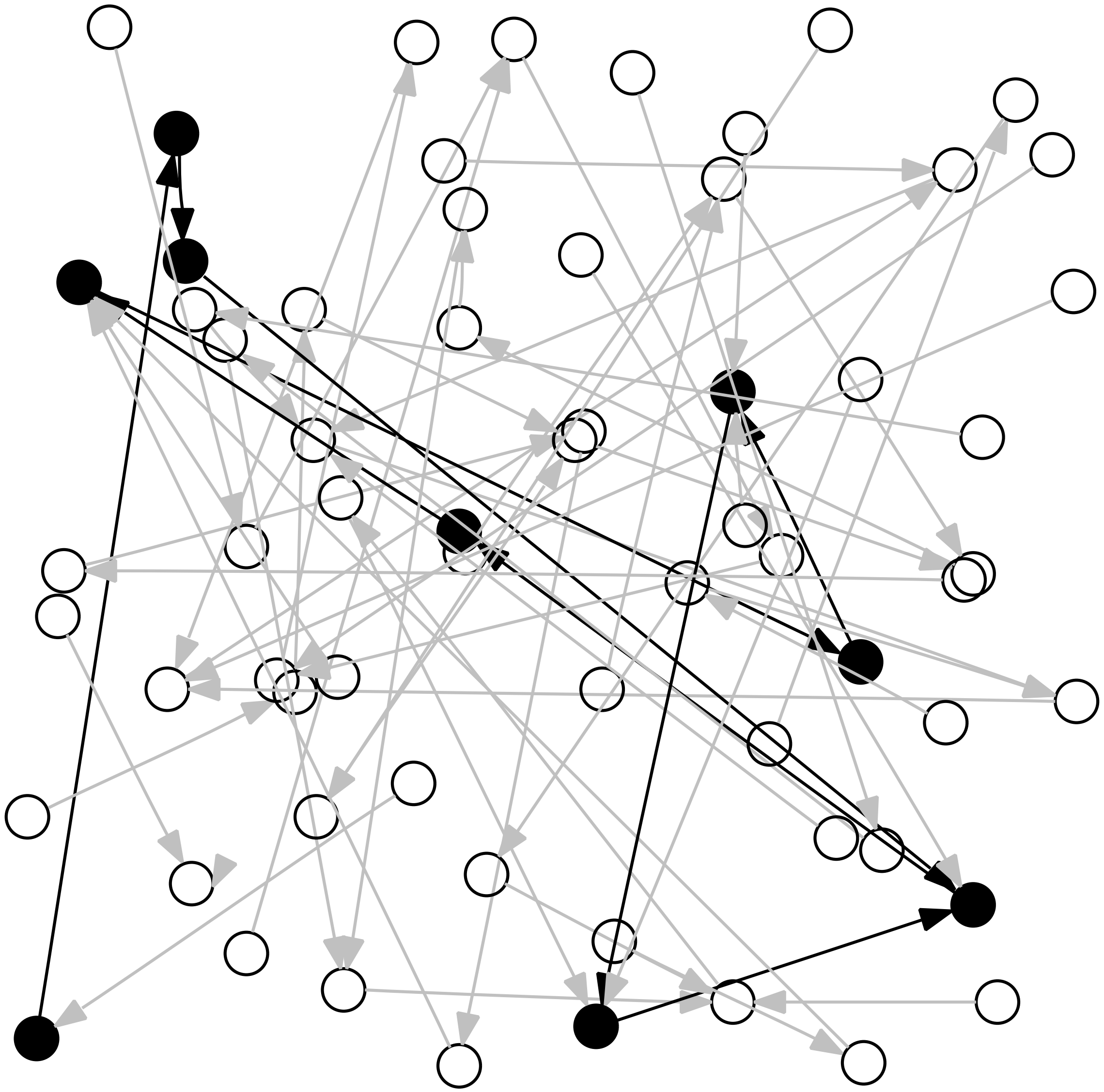

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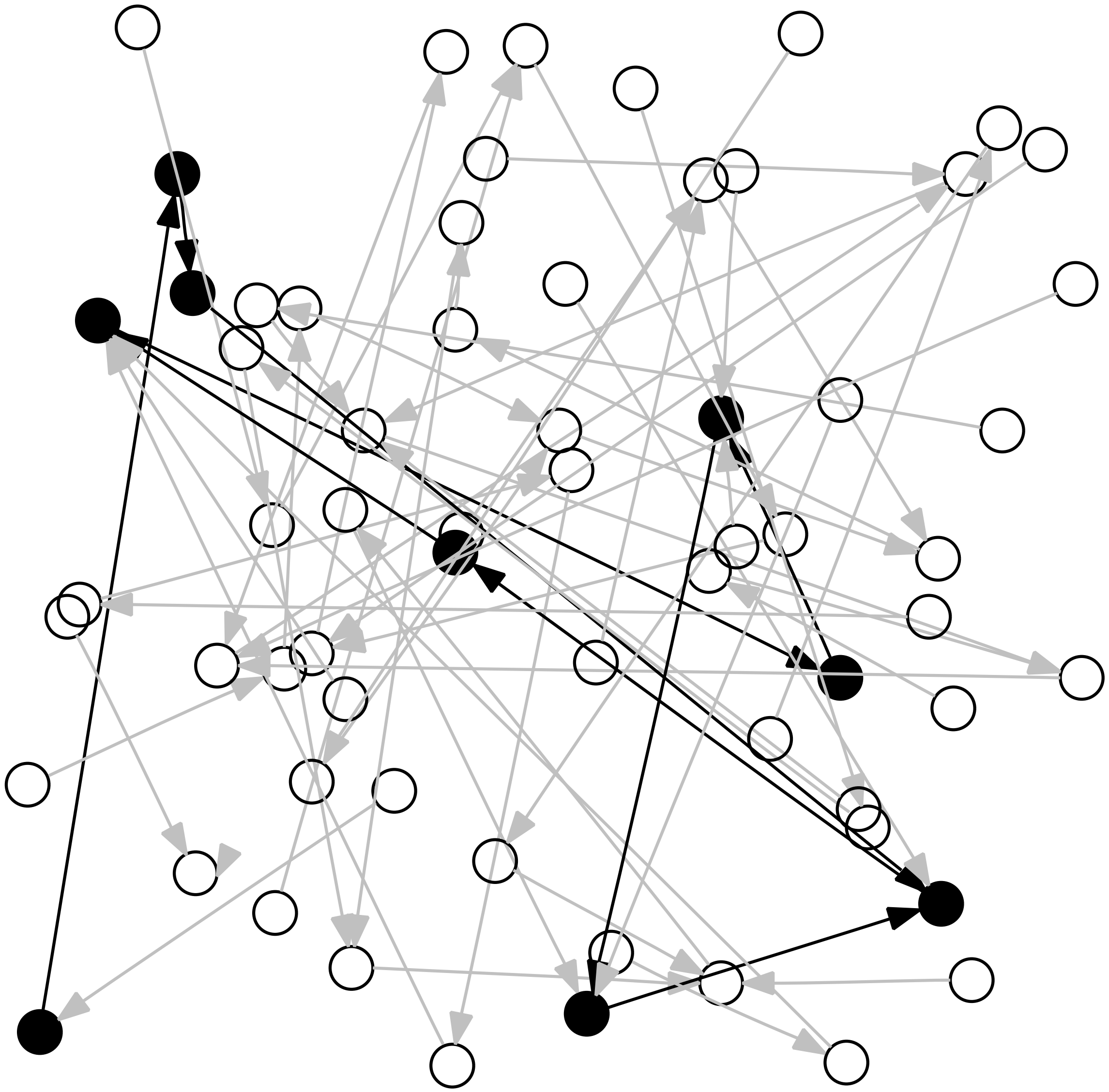

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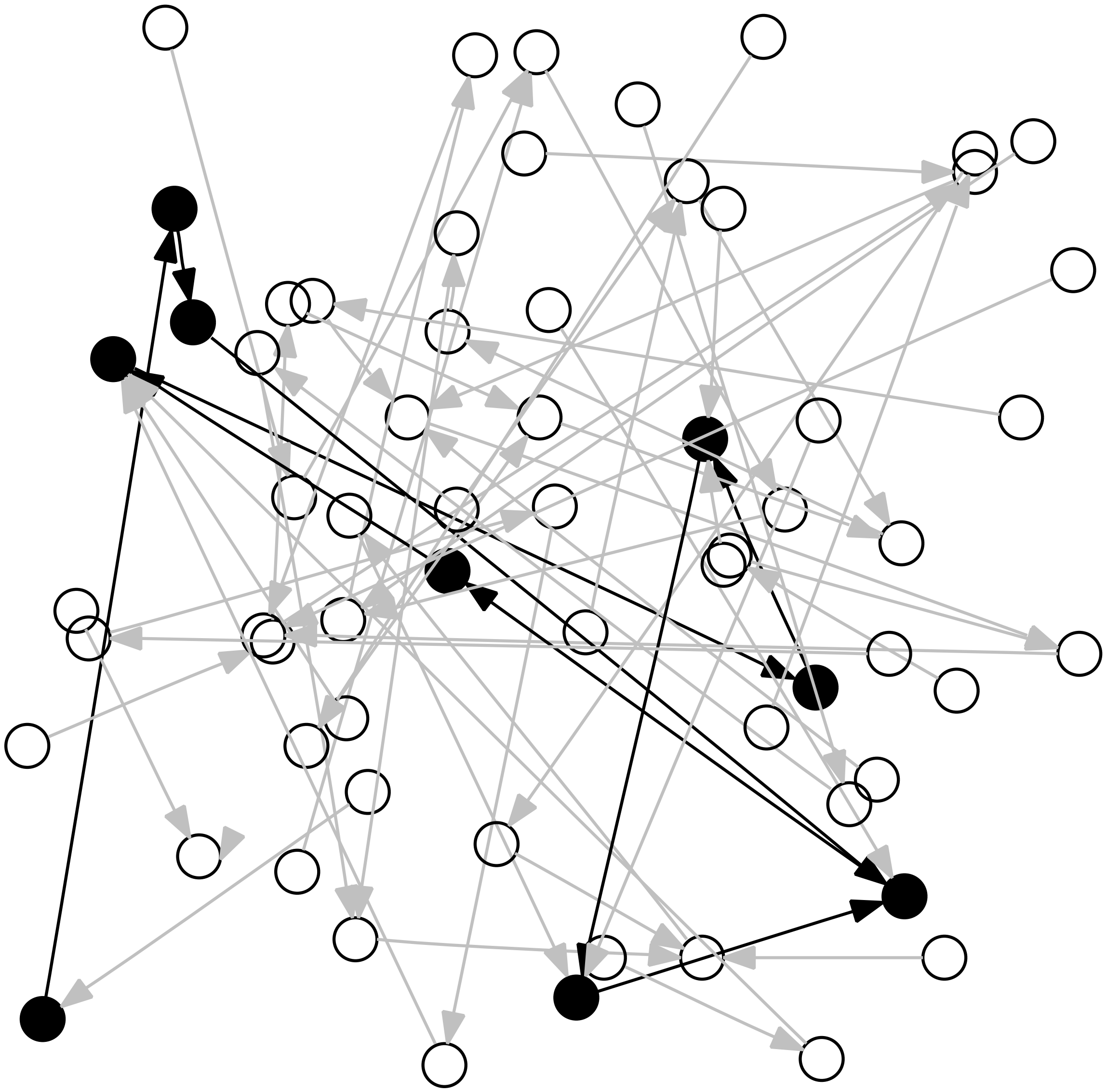

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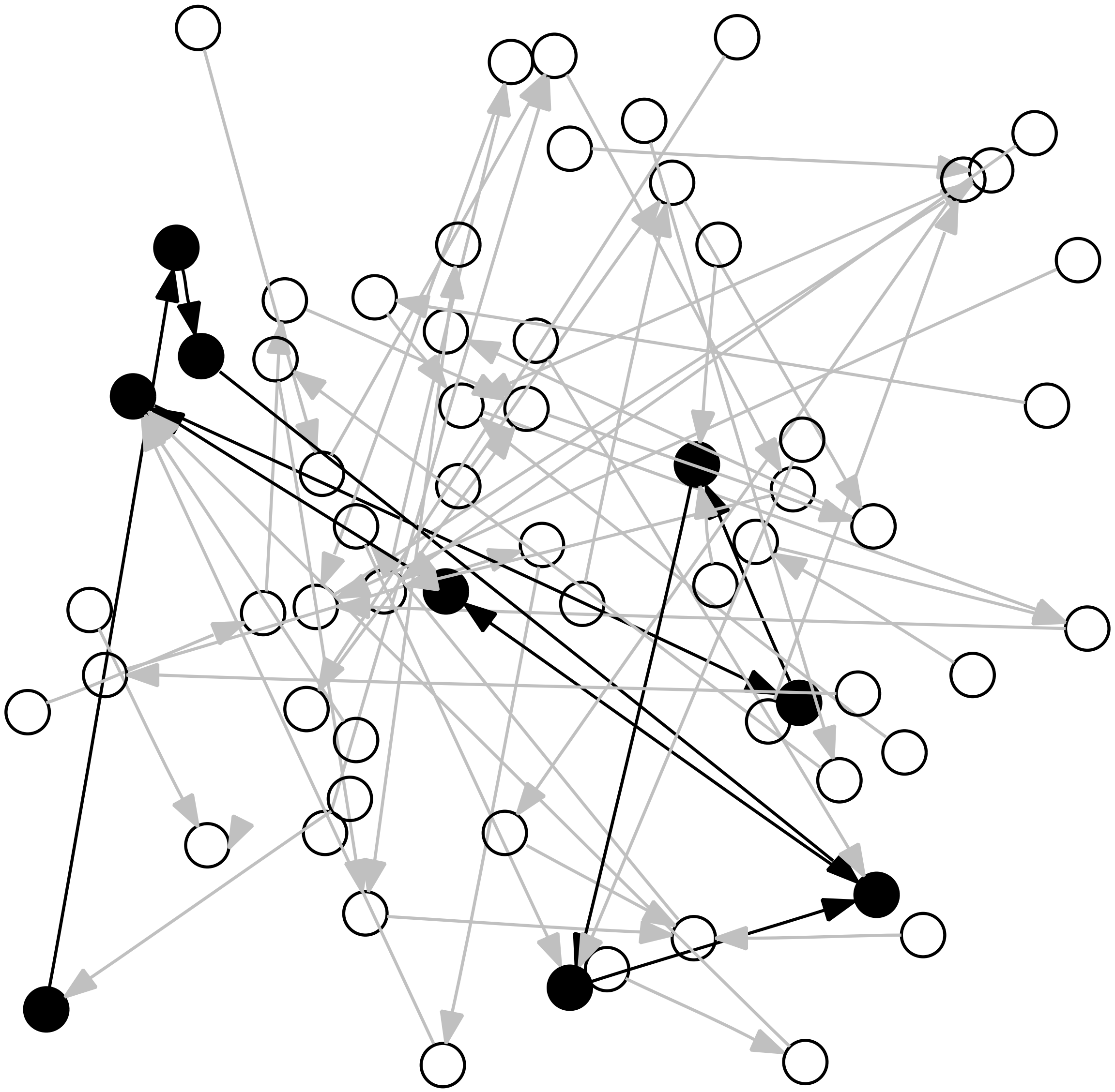

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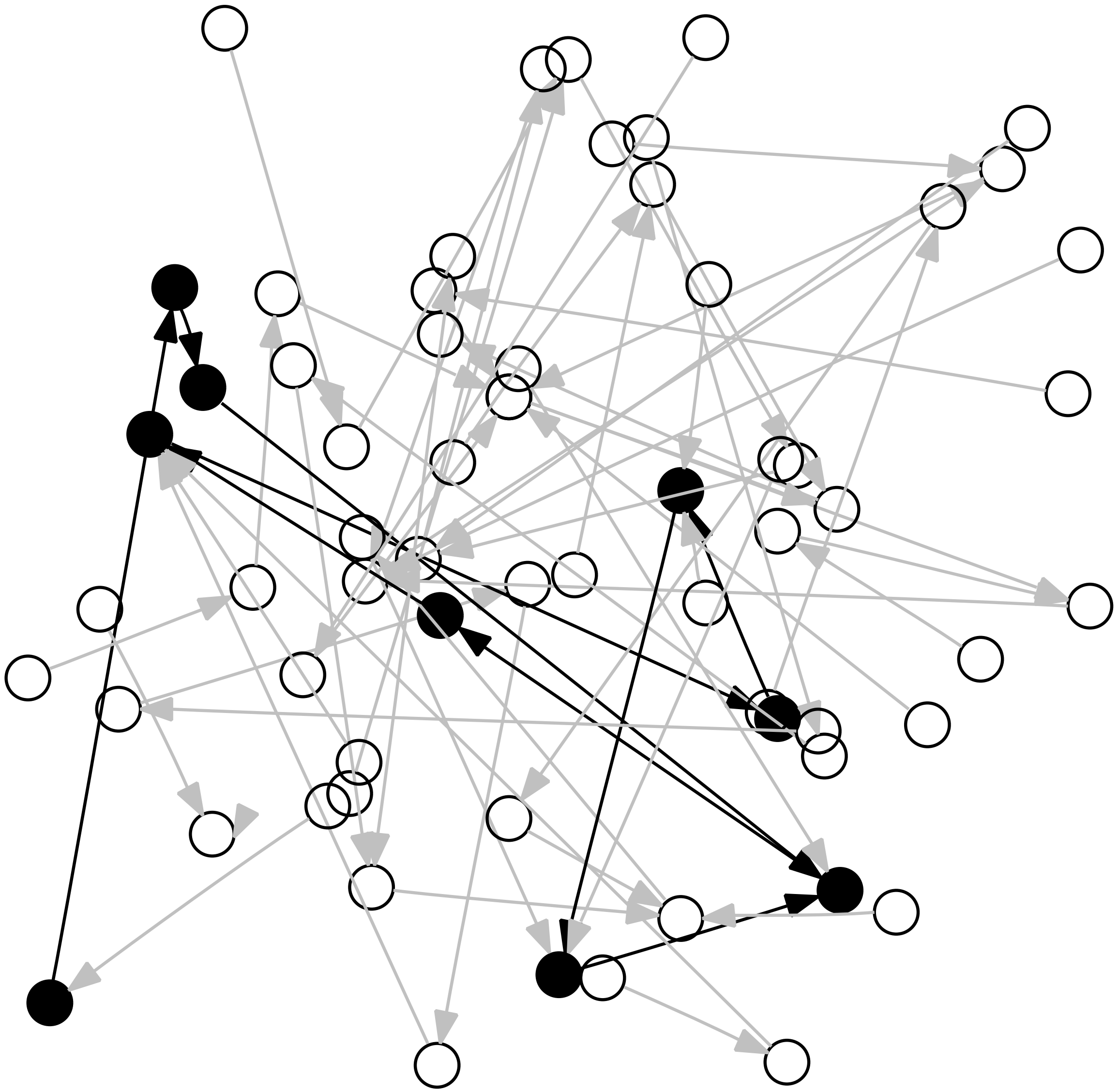

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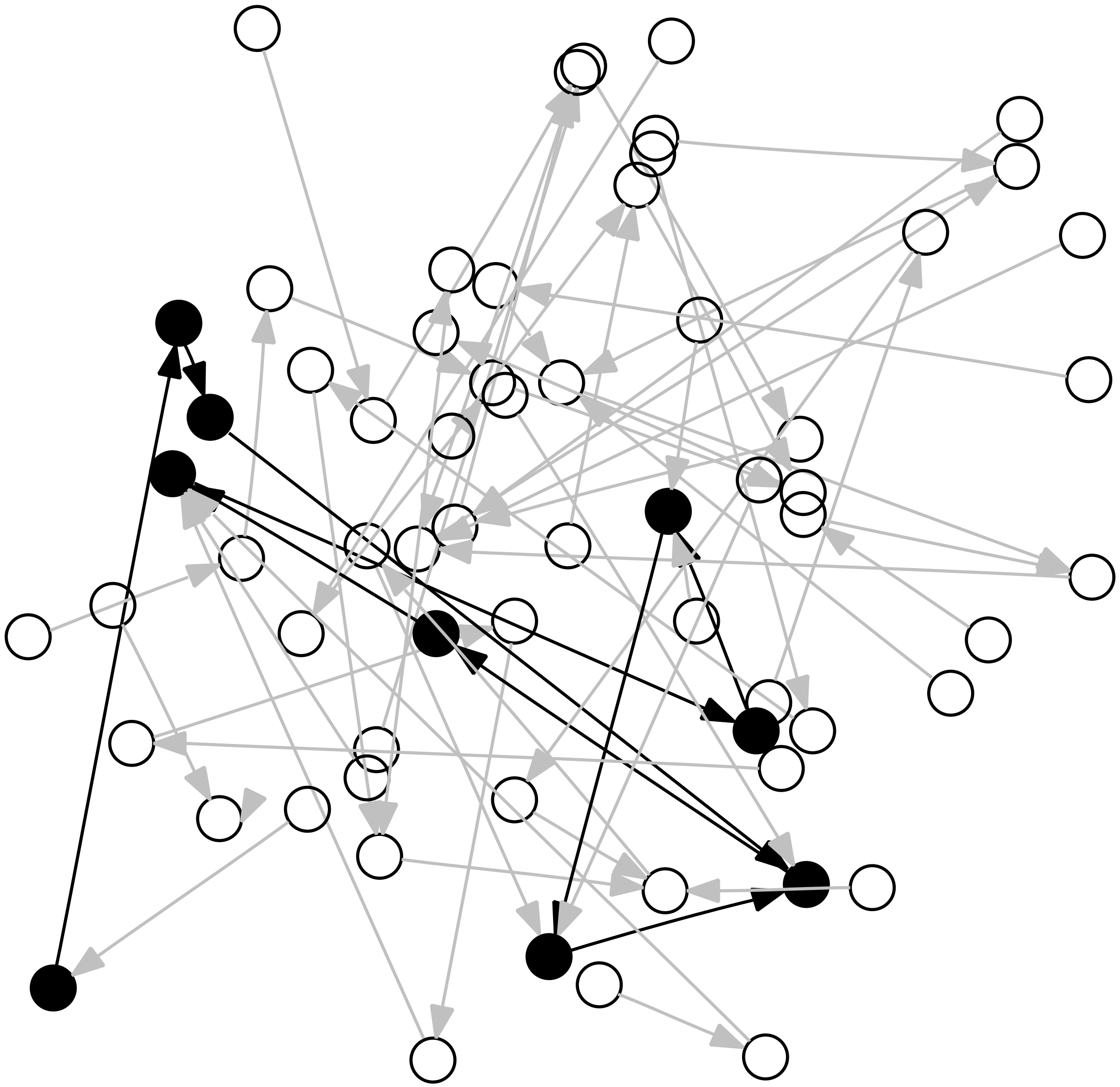

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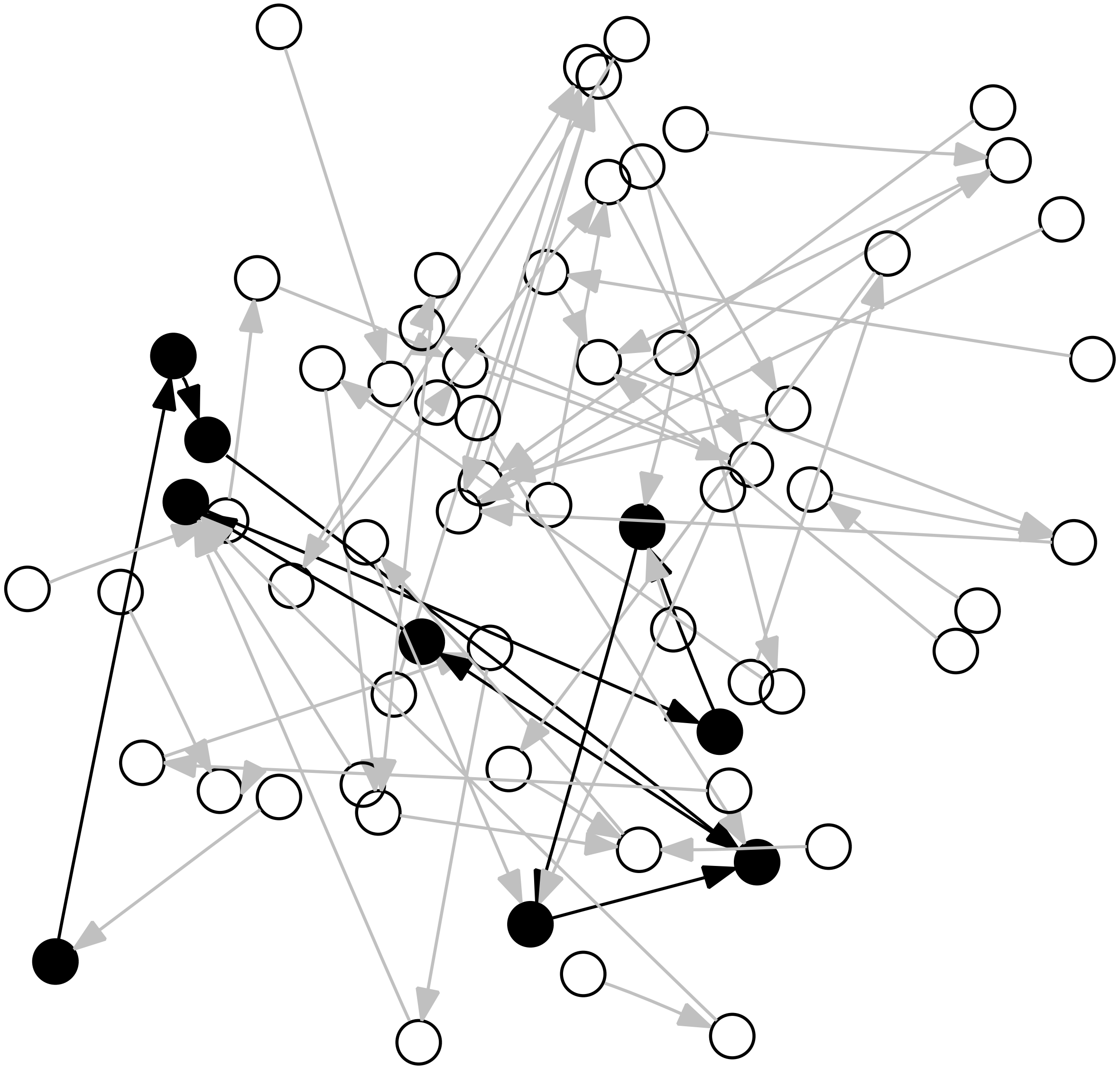

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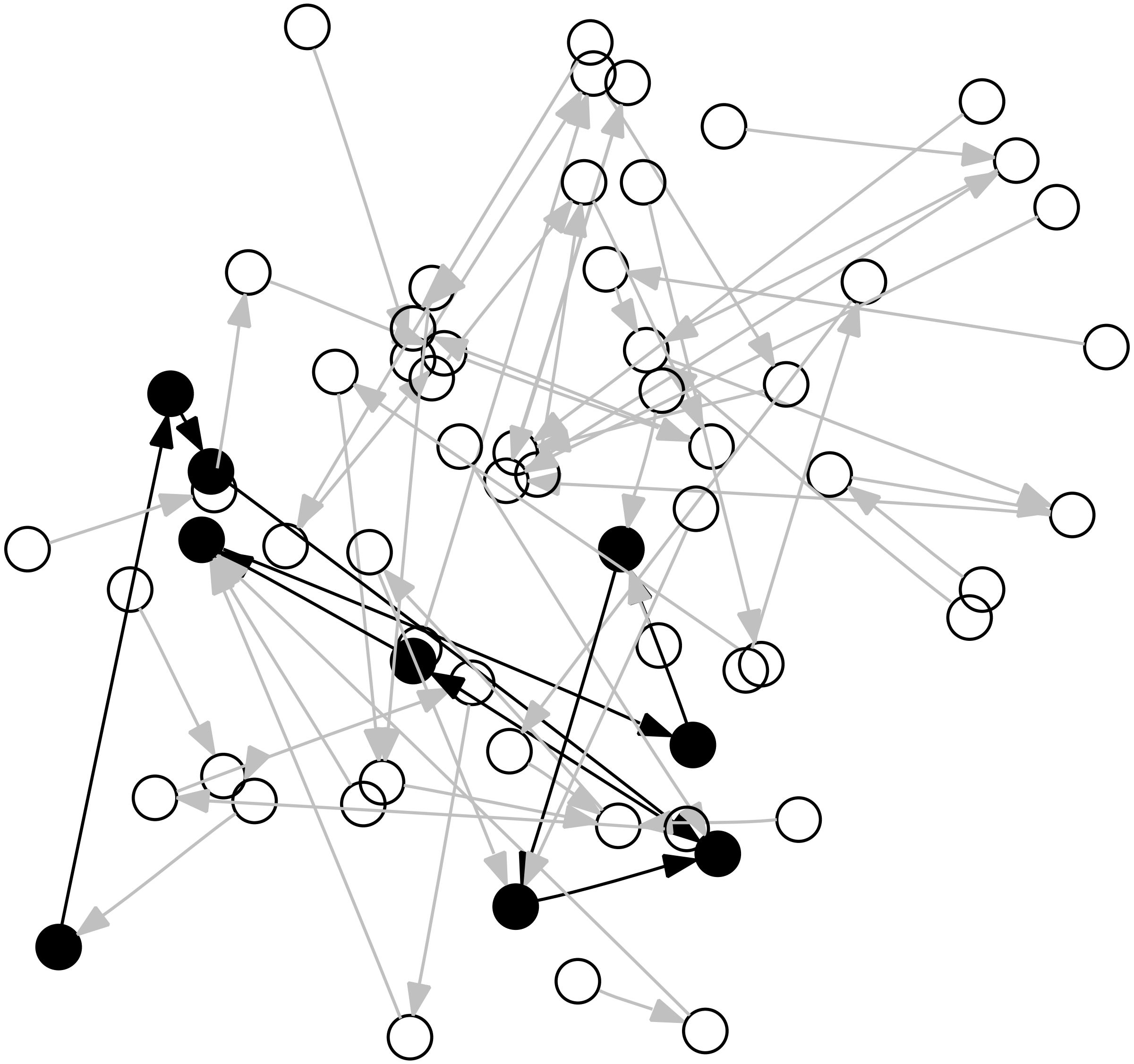

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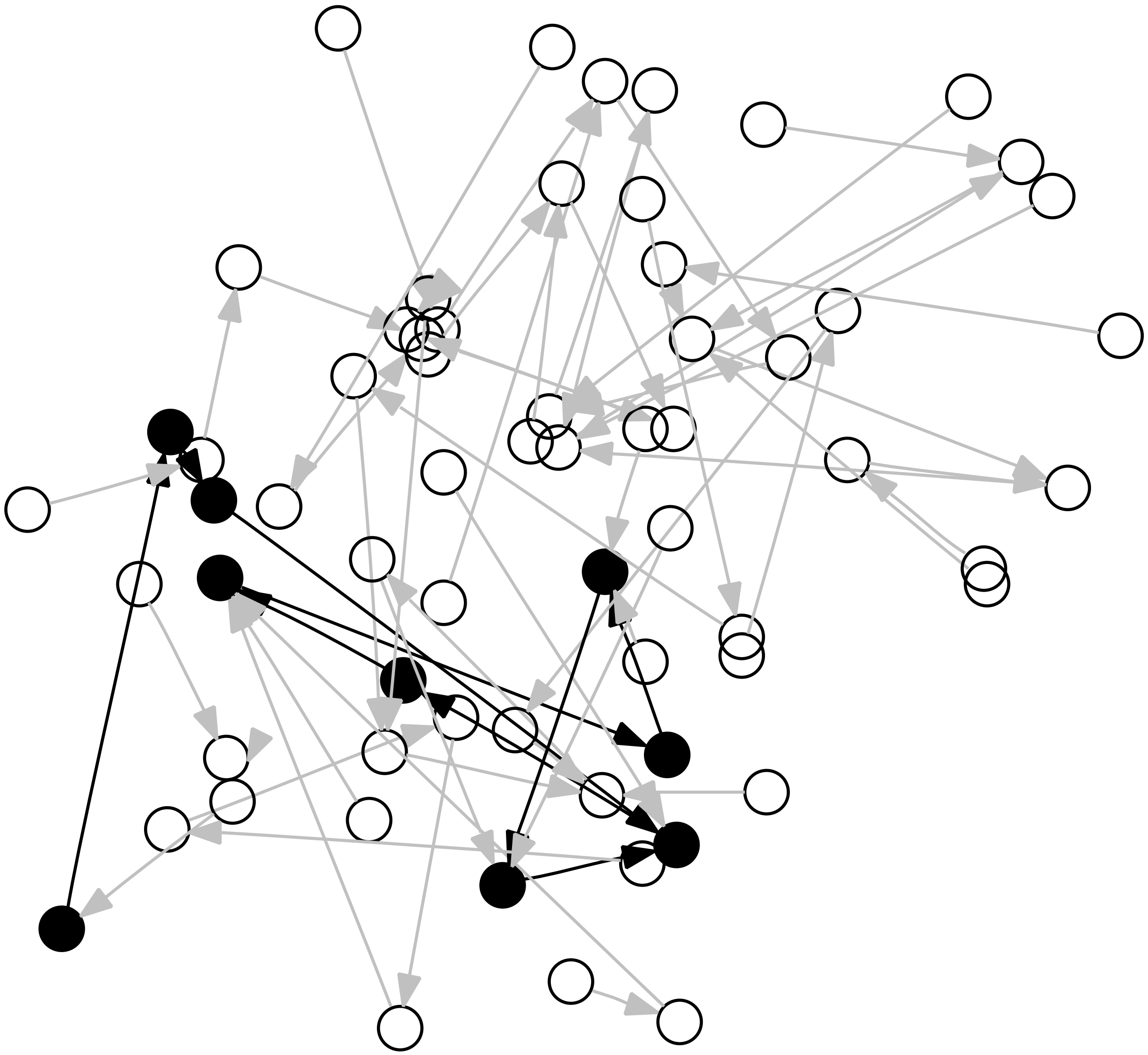

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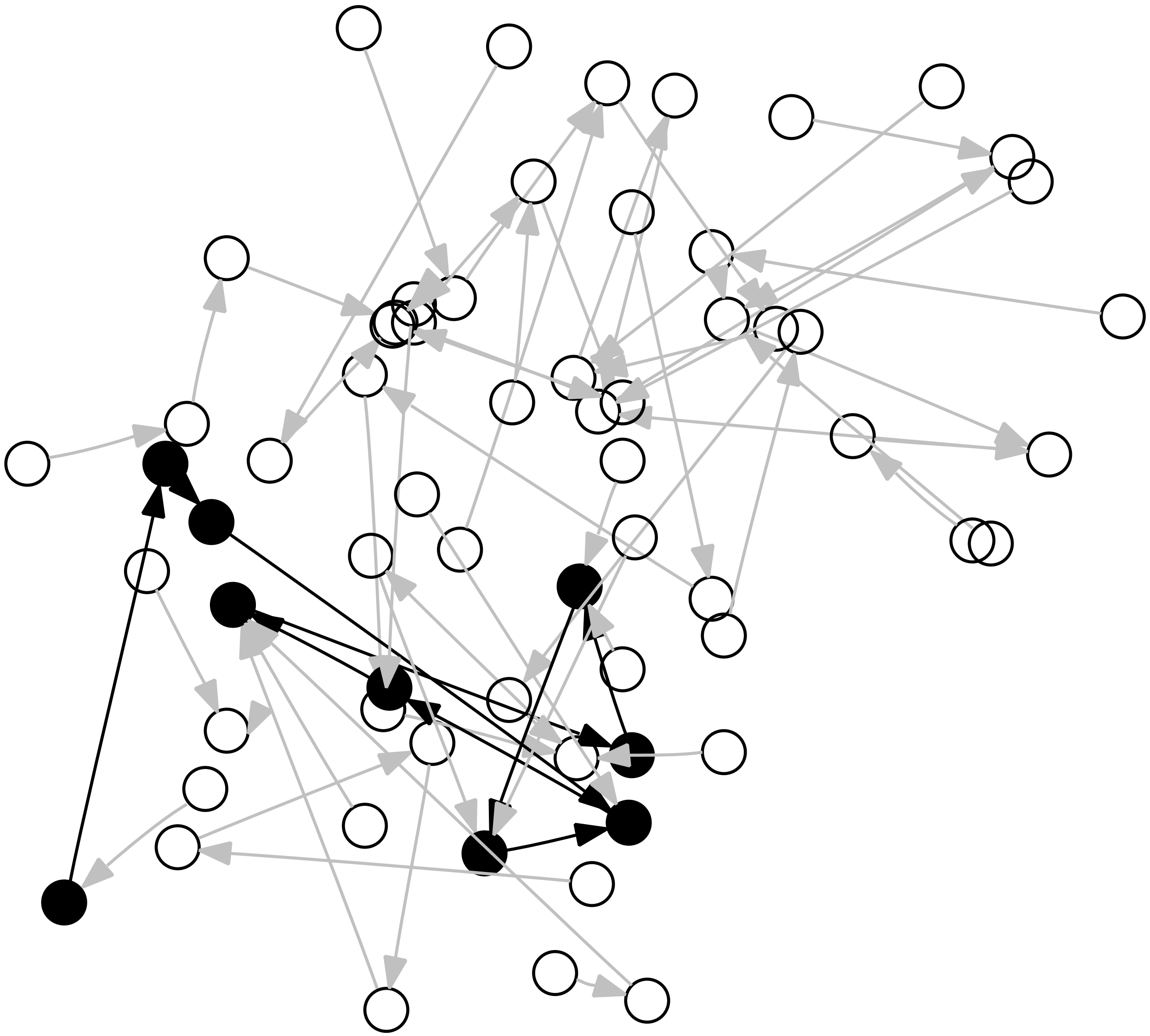

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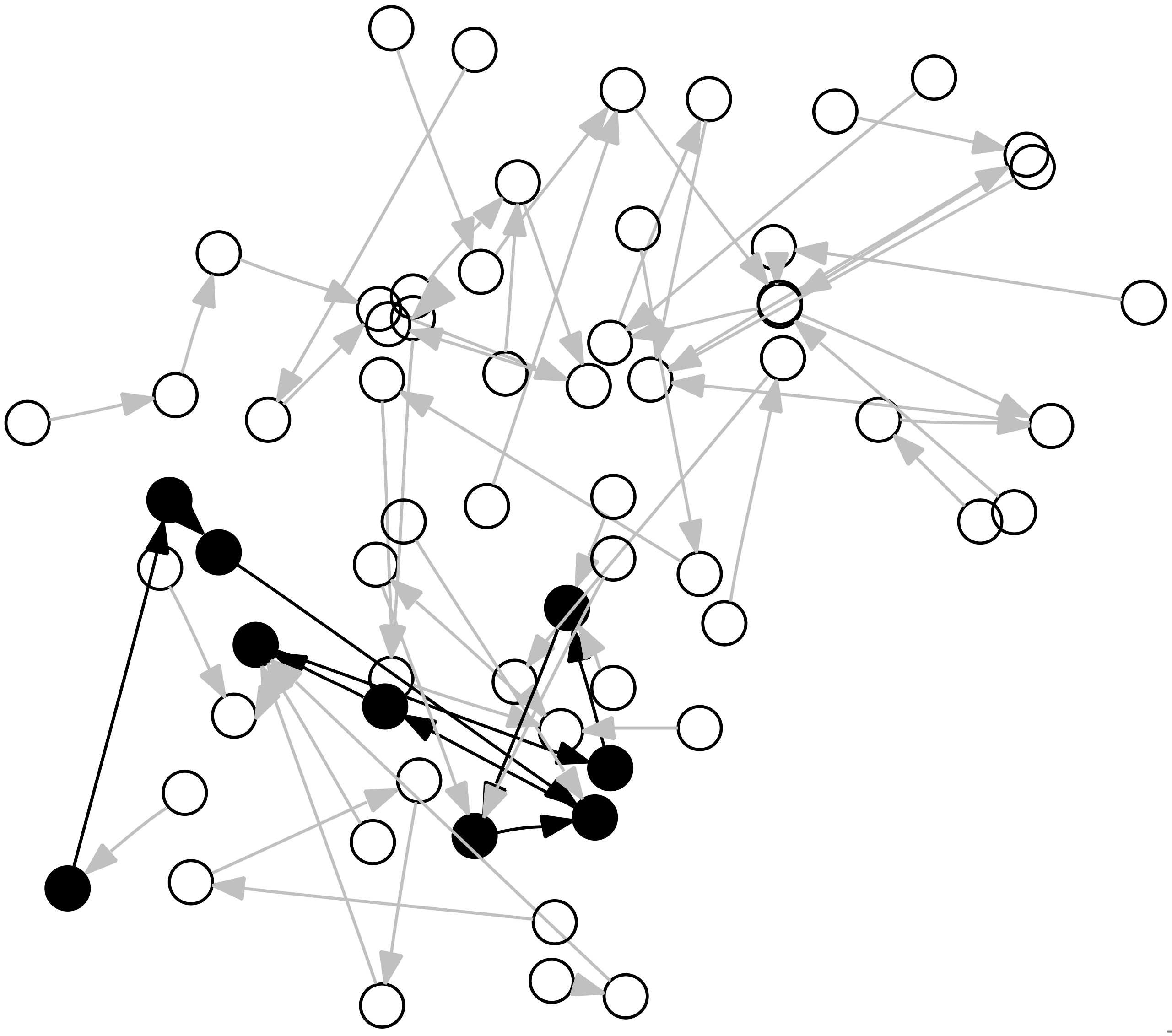

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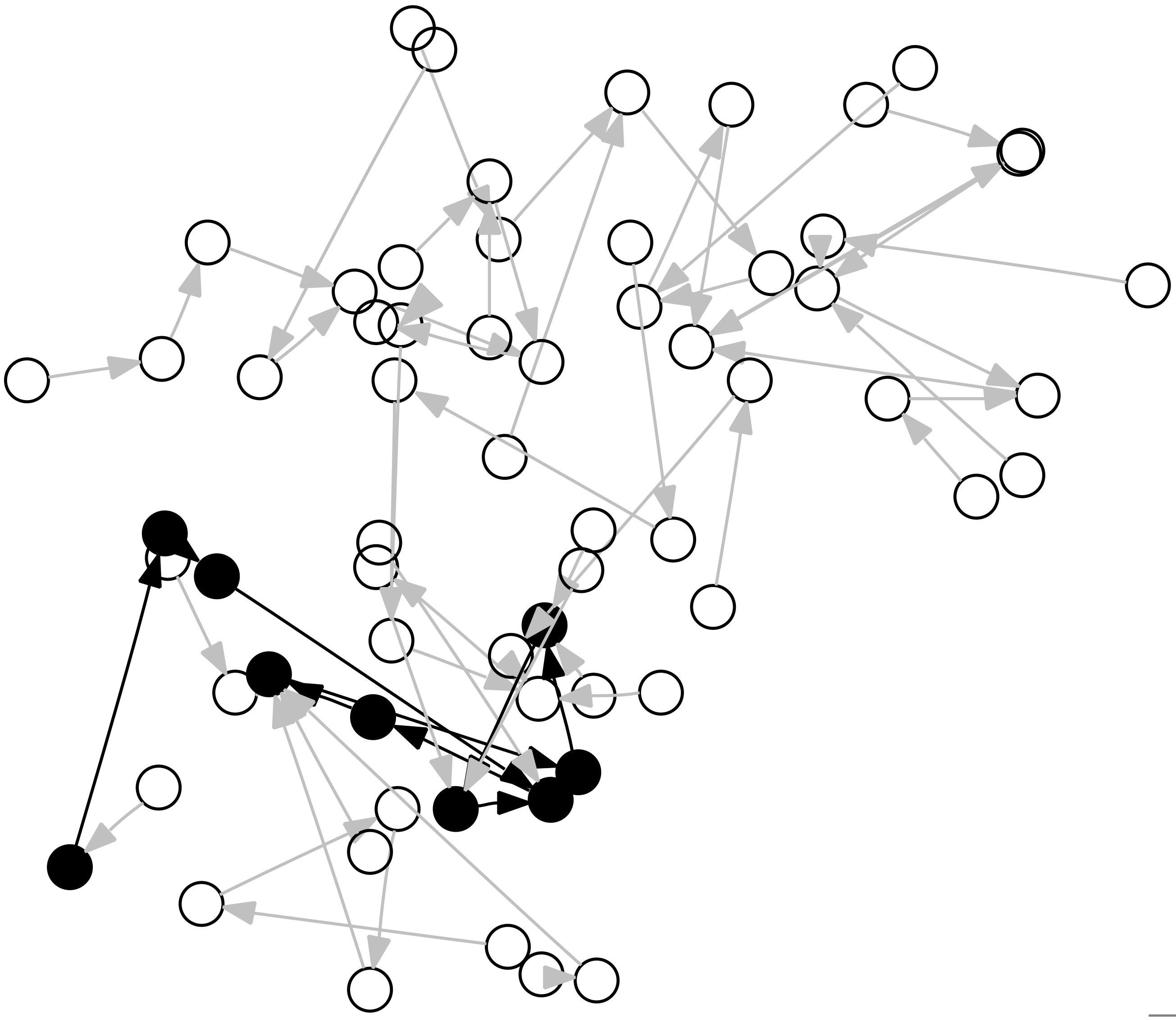

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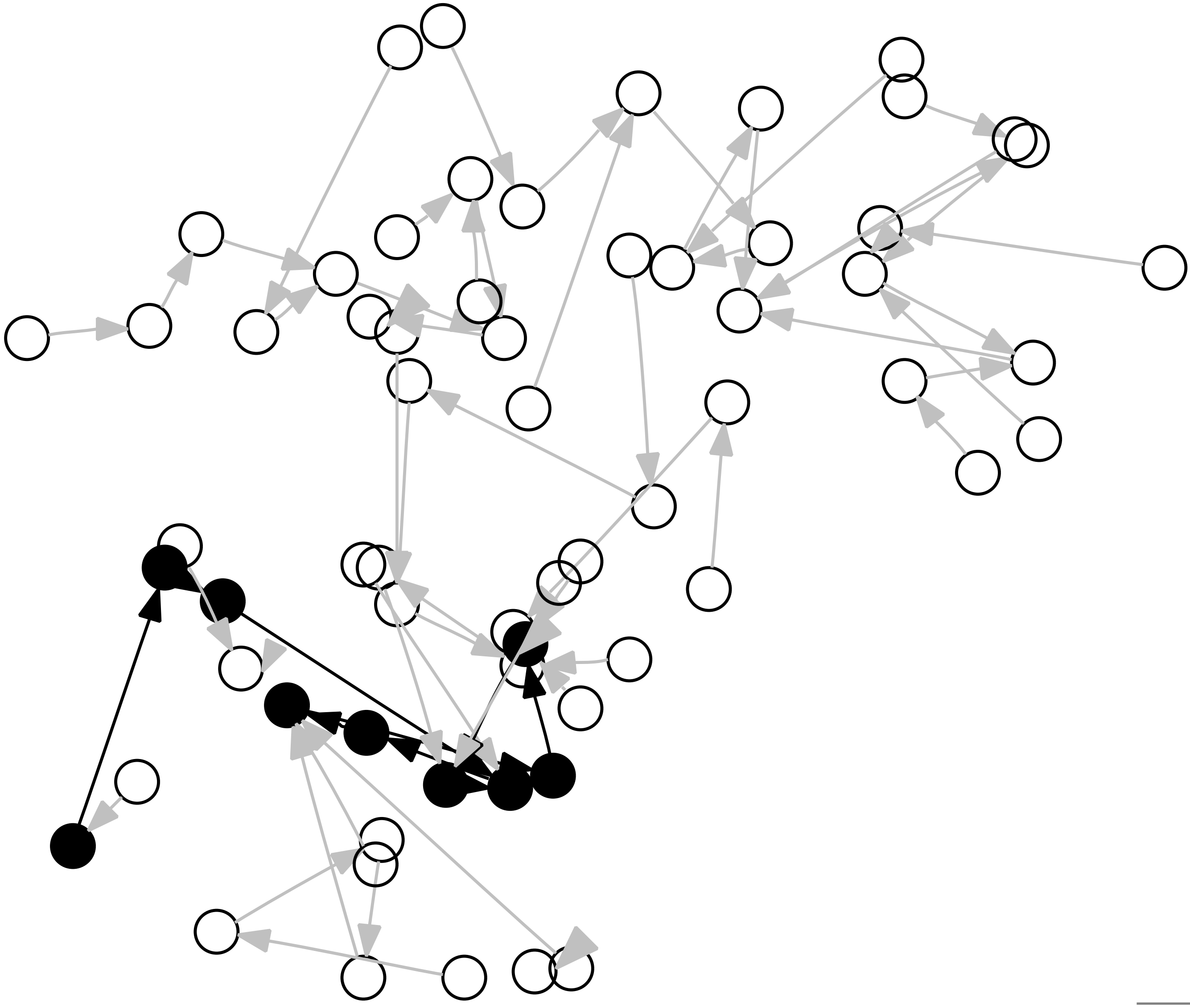

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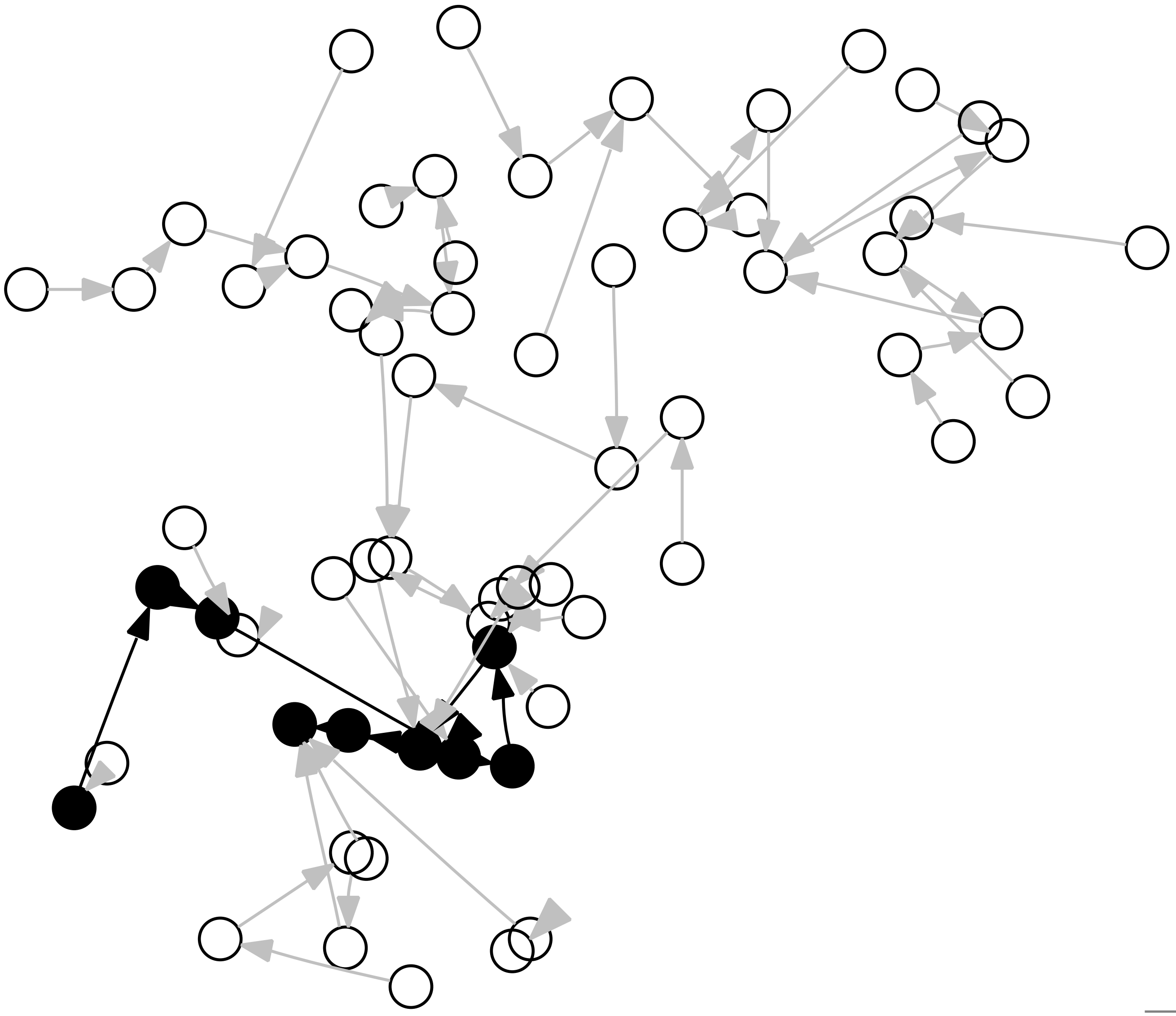

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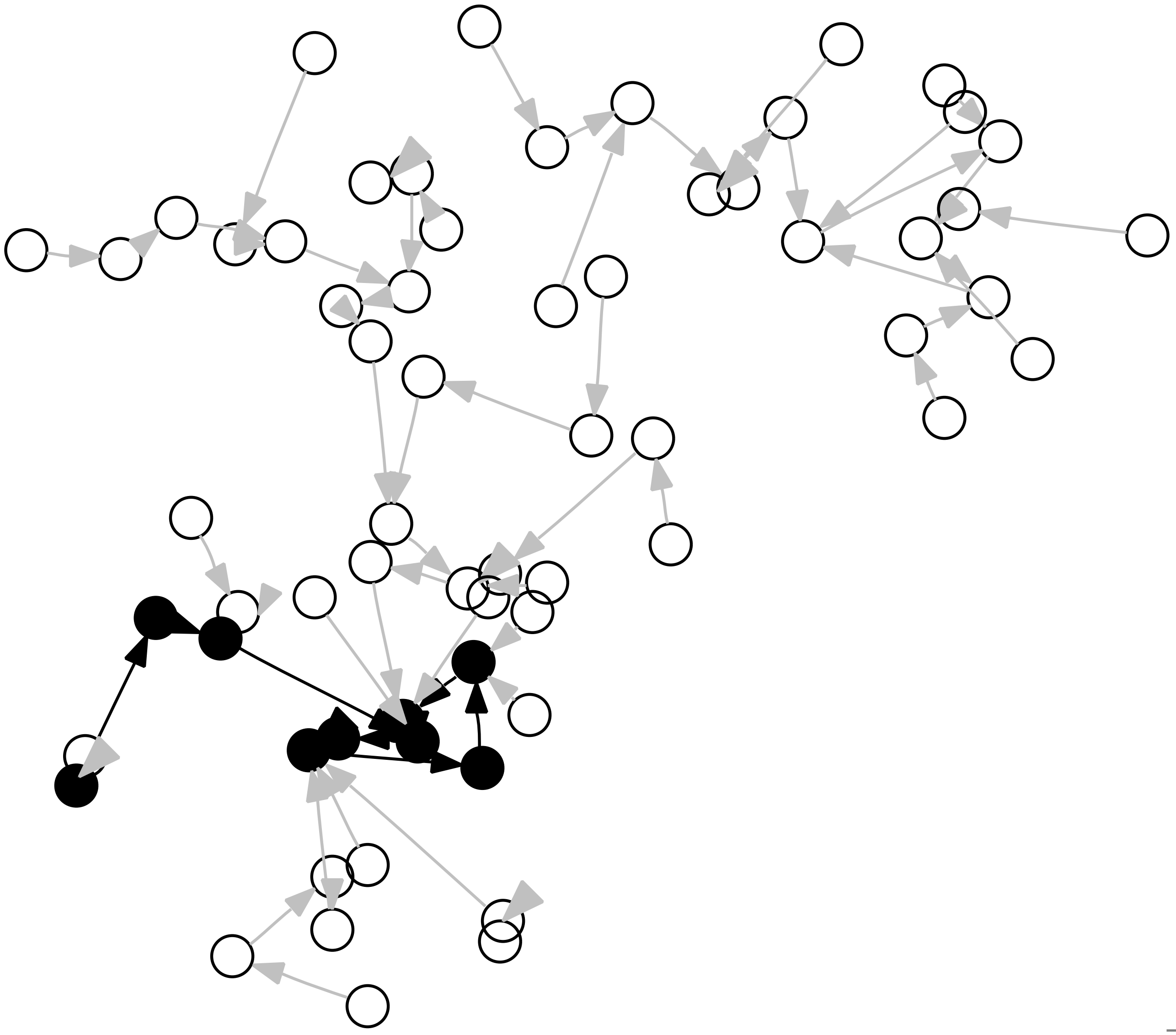

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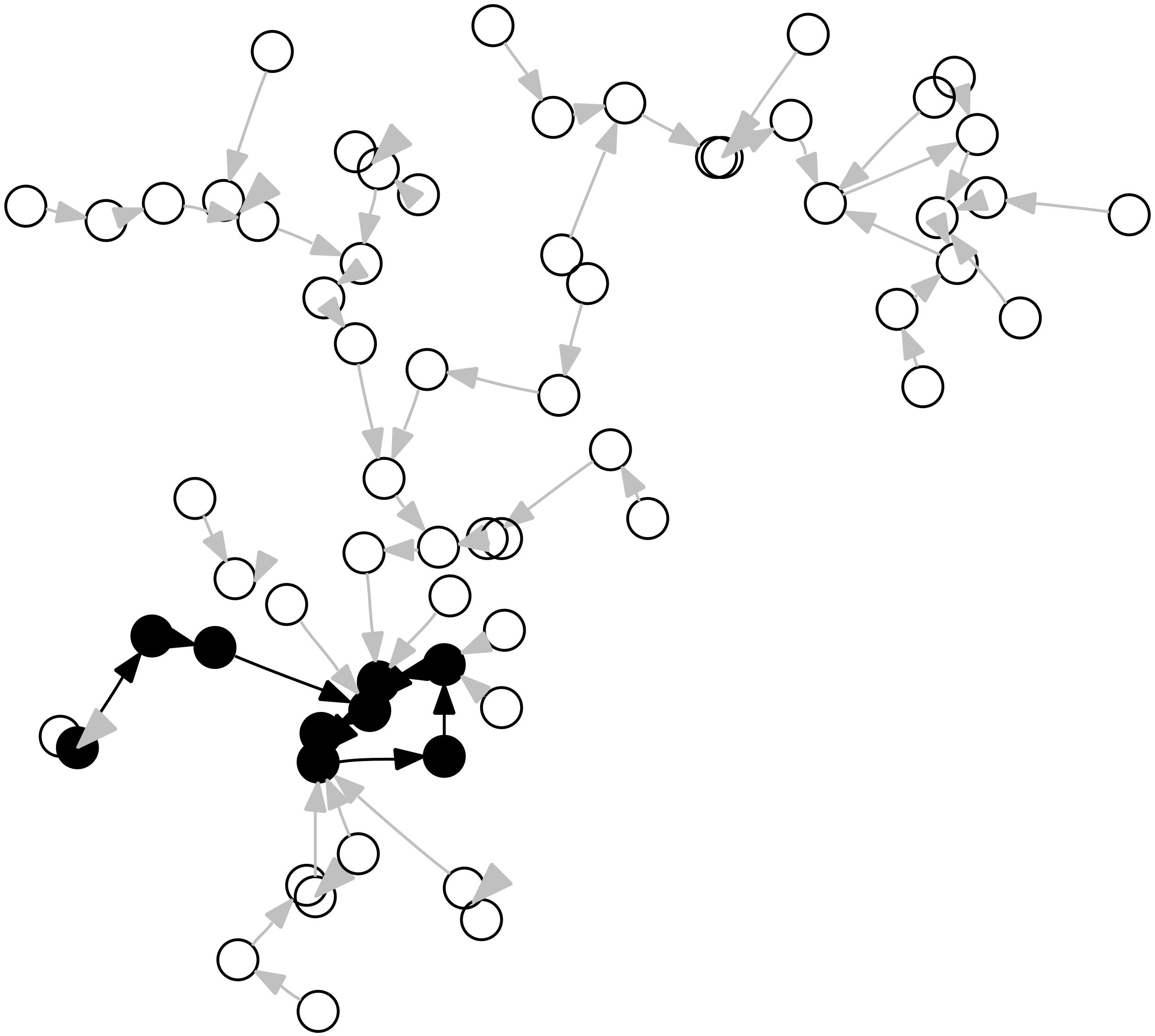

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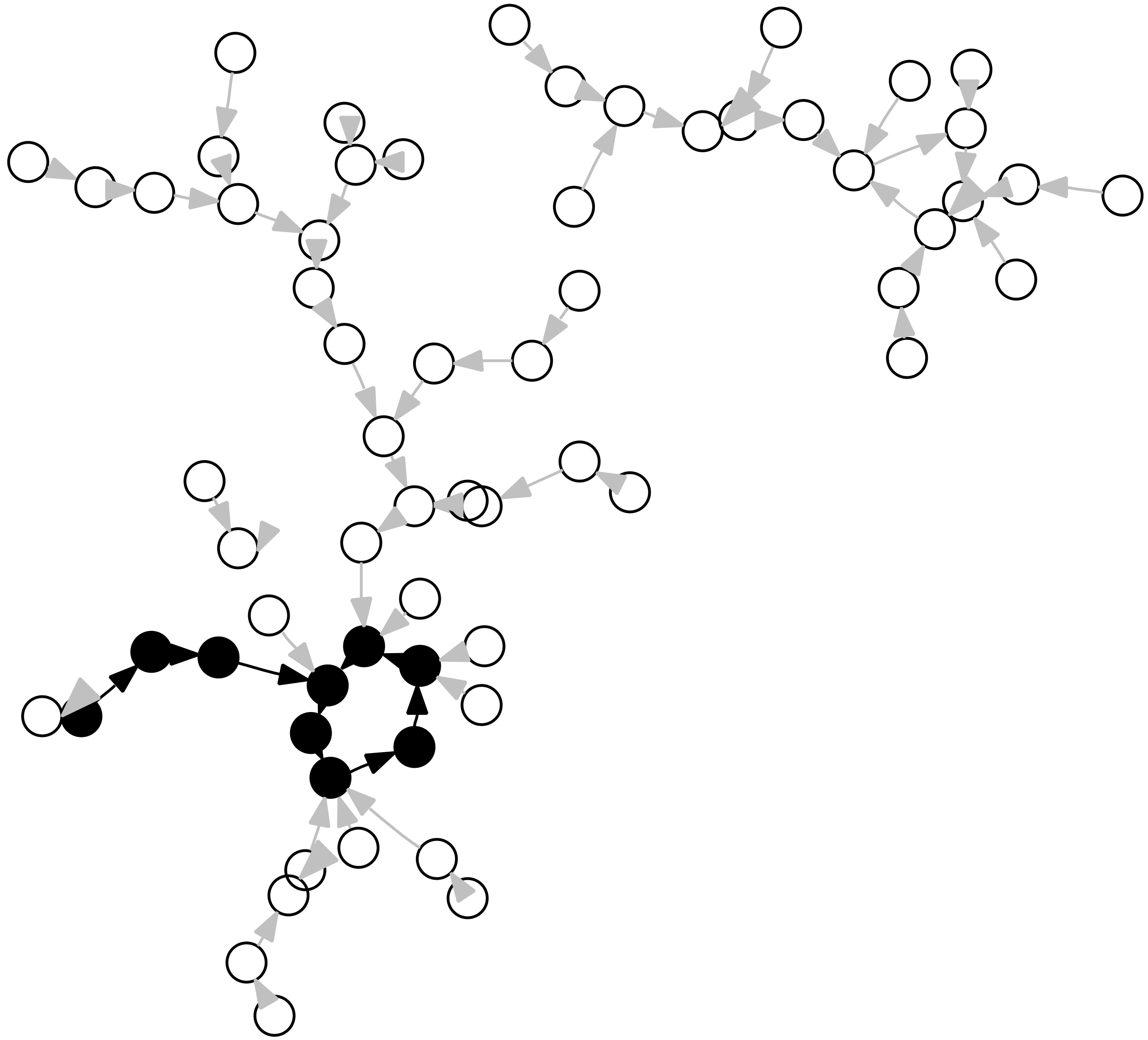

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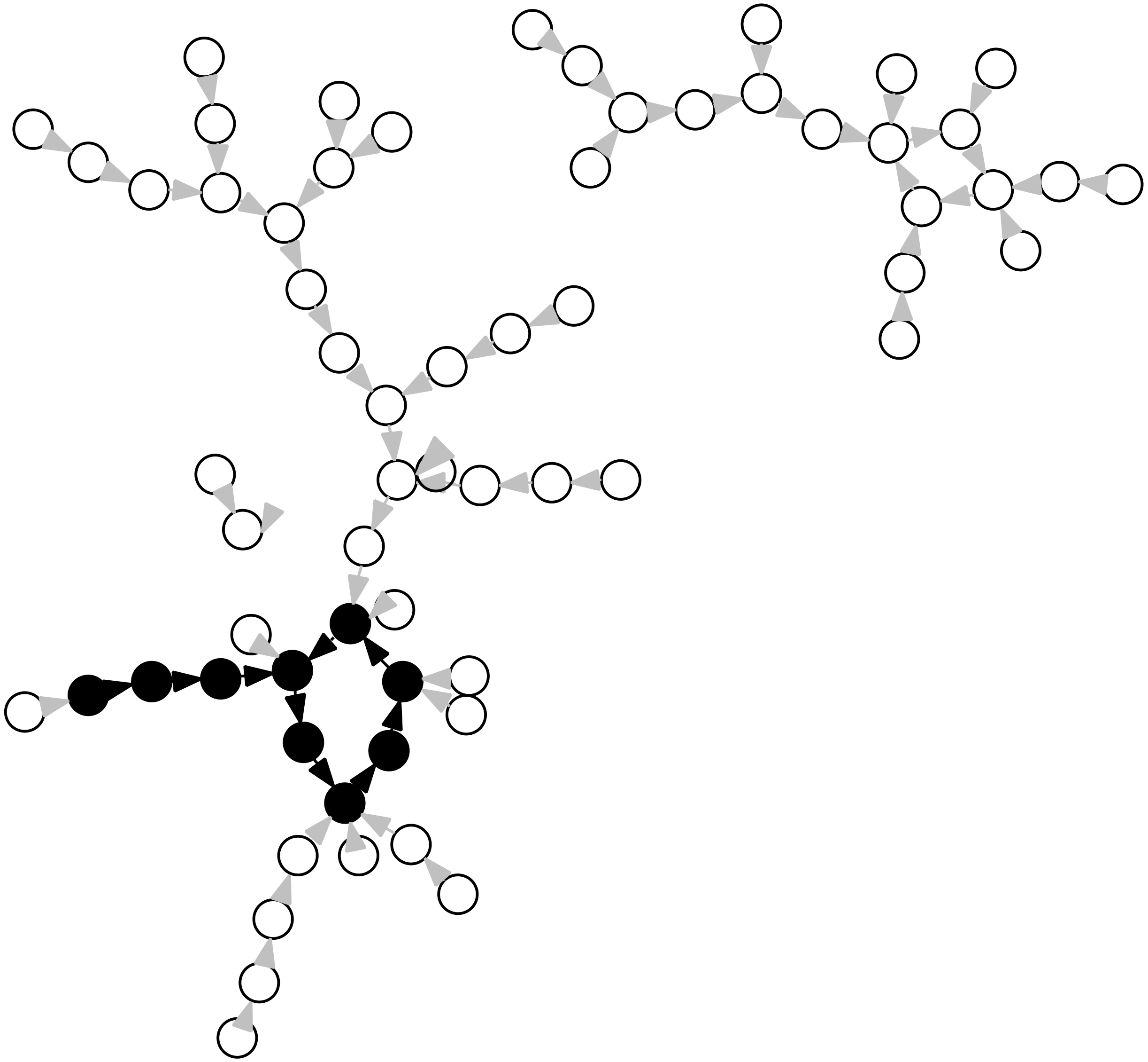


Connection with DLP

- Assume that for each point we know $a_i, b_i \in \mathbb{Z}/\ell\mathbb{Z}$ so that $W_i = a_i P + b_i Q$.
- Then $W_i = W_j$ means that

$$a_i P + b_i Q = a_j P + b_j Q$$

so $(b_i - b_j)Q = (a_j - a_i)P$. If $b_i \neq b_j$ the DLP is solved:

$$a = (a_j - a_i)/(b_i - b_j).$$

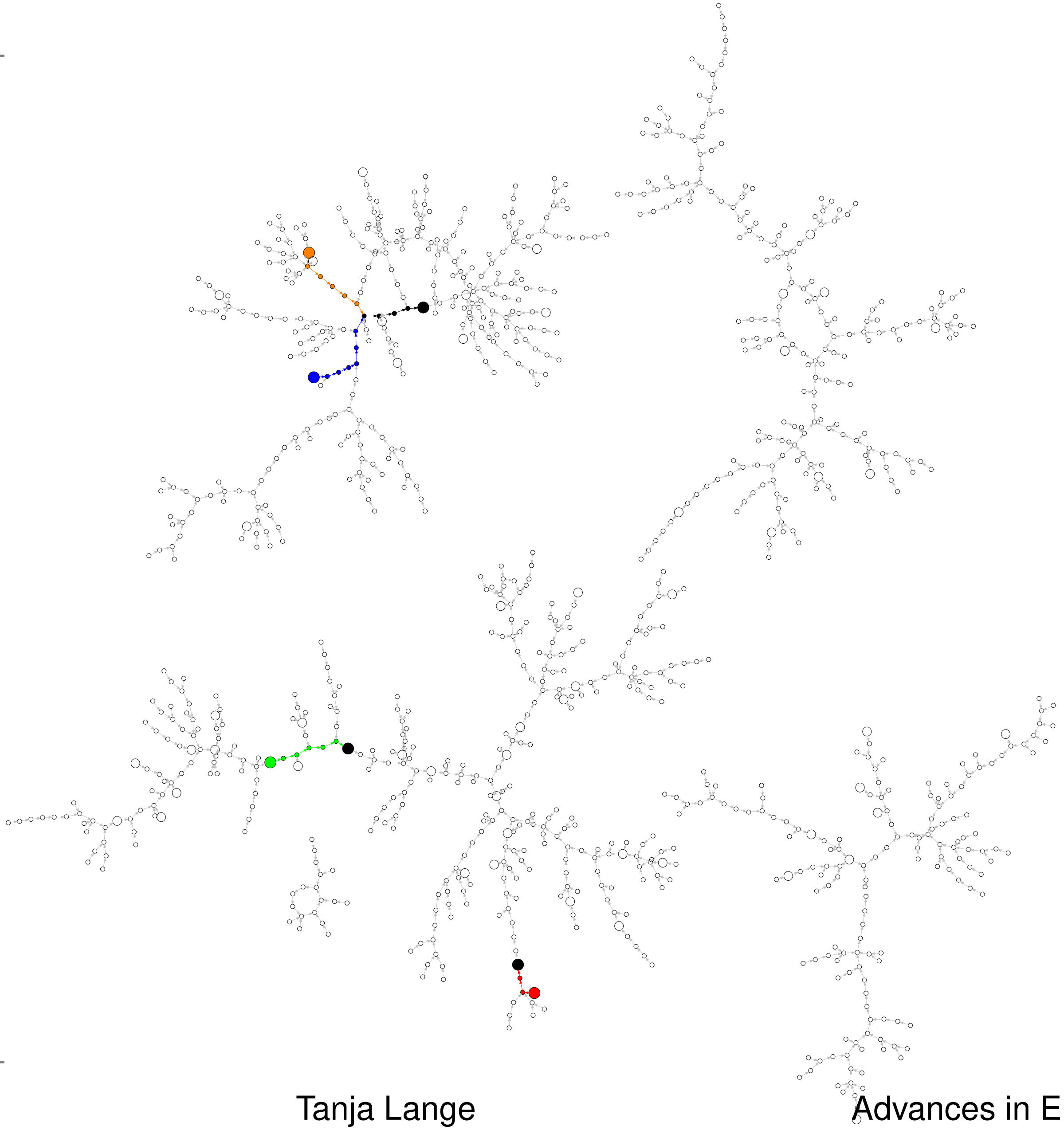
- e.g. additive walk: Start with $W_0 = P$ and put

$$f(W_i) = W_i + c_j P + d_j Q,$$

where $j = h(W_i)$.

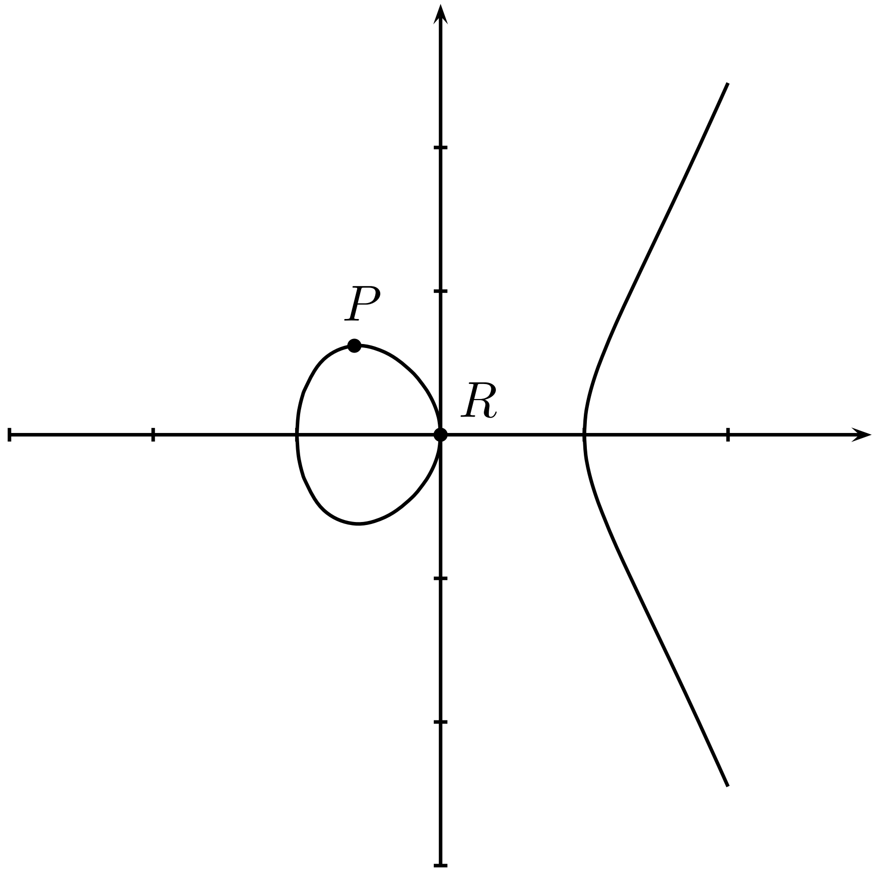
Parallel rho

- Perform many walks with different starting points but same update function f.
- If two different walks find the same point then their subsequent steps will match.
- Terminate each walk once it hits a distinguished point.
- Attacker chooses frequency and definition of distinguished points. Do not wait for cycle.
- Collect all distinguished points. Two walks ending in same distinguished point solve DLP.



Group law in $E(\mathbb{R})$

$y^2 = x^3 - x$



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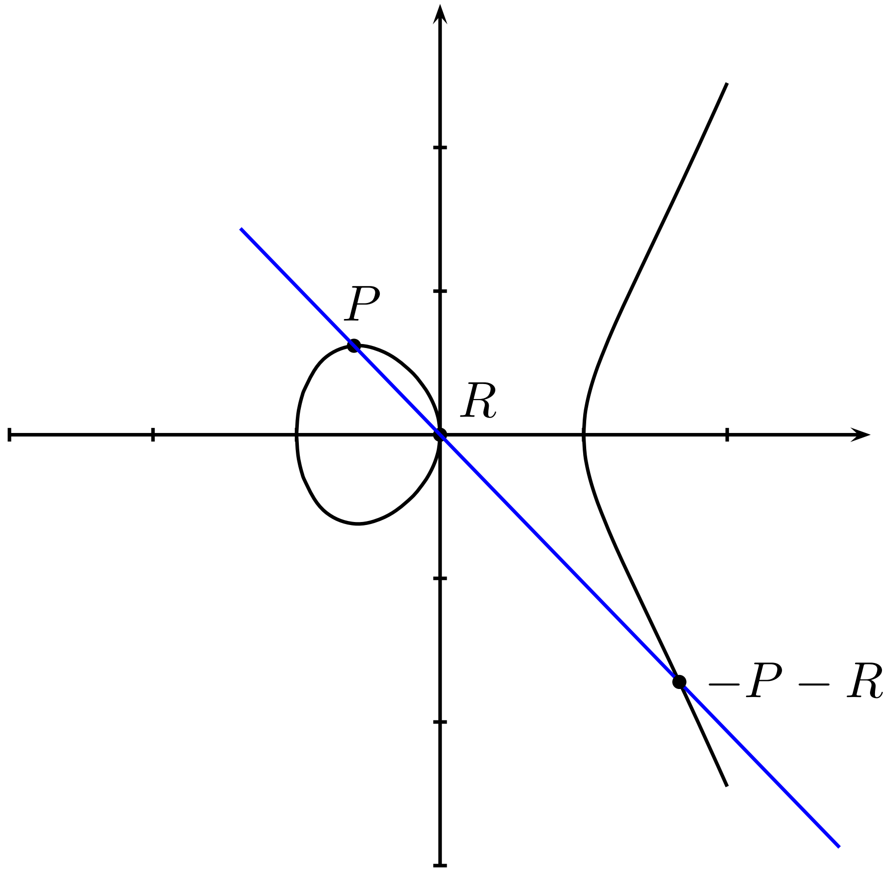

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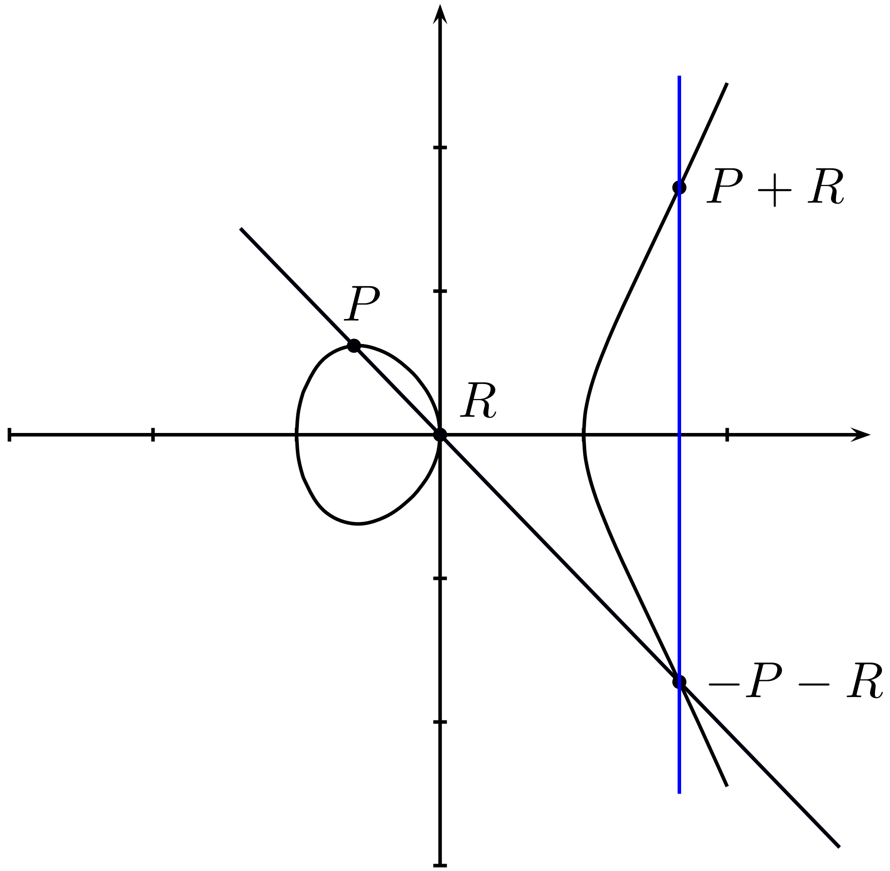
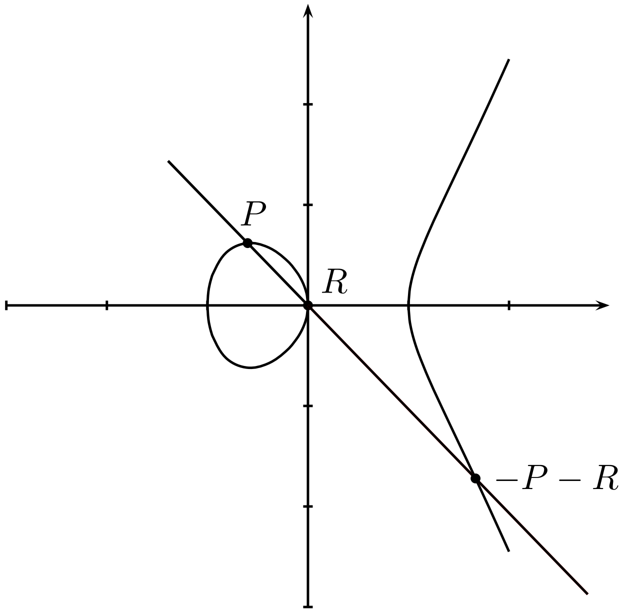


Group law in formulas

$$E : y^2 = x^3 + a_4 x + a_6,\ a_i \in \mathbb{F}_q$$



Line $y = \lambda x + \mu$ has slope

$$\lambda = \frac{y_R - y_P}{x_R - x_P}.$$

Equating gives

$$(\lambda x + \mu)^2 = x^3 + a_4 x + a_6.$$

This equation has 3 solutions, the x-coordinates of P, R and $-P-R$, thus

$$\begin{aligned}(x - x_P)(x - x_R)(x - x_{-P-R}) &= x^3 - \lambda^2 x^2 + (a_4 - 2\lambda\mu)x + a_6 - \mu^2 \\ x_{-P-R} &= \lambda^2 - x_P - x_R\end{aligned}$$

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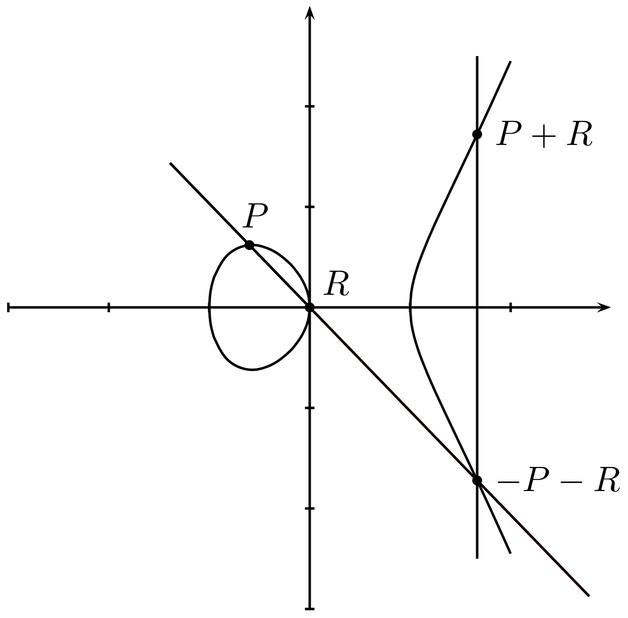

Point P is on line, thus

$$y_P = \lambda x_P + \mu, \text{ i.e.}$$
$$\mu = y_P - \lambda x_P,$$

and

$$\begin{aligned} y_{-P-R} &= \lambda x_{-P-R} + \mu \\ &= \lambda x_{-P-R} + y_P - \lambda x_P \\ &= \lambda(x_{-P-R} - x_P) + y_P \end{aligned}$$

Point $P + R$ has the same x-coordinate but negative y-coordinate:

$$x_{P+R} = \lambda^2 - x_P - x_R, \quad y_{P+R} = \lambda(x_P - x_{P+R}) - y_P$$

Group law in formulas

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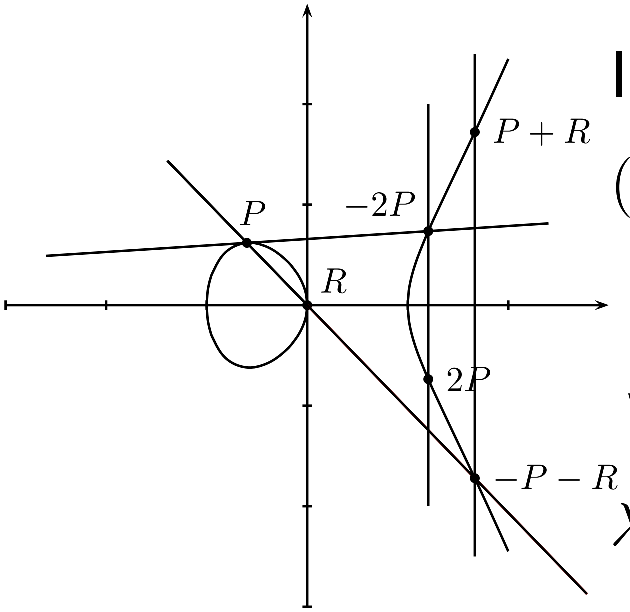

In general, for $(x_P, y_P) \neq (x_R, -y_R)$:

$$\begin{aligned}(x_P, y_P) + (x_R, y_R) &= \\ &= (x_{P+R}, y_{P+R}) = \\ &= (\lambda^2 - x_P - x_R, \lambda(x_P - x_{P+R}) - y_P),\end{aligned}$$

where

$$\lambda = \begin{cases} (y_R - y_P)/(x_R - x_P) & \text{if } x_P \neq x_R, \\ (3x_P^2 + a_4)/(2y_P) & \text{else.} \end{cases}$$

$\Rightarrow$ Addition and Doubling need
1 I, 2M, 1S and 1 I, 2M, 2S, respectively

Negation and rho

- $W = (x, y)$ and $-W = (x, -y)$ have same x-coordinate. Search for x-coordinate collision.
- Search space for collisions is only $\lceil \ell/2 \rceil$; this gives factor $\sqrt{2}$ speedup … if $f(W_i) = f(-W_i)$.
- To ensure $f(W_i) = f(-W_i)$ define $j = h(|W_i|)$ and $f(W_i) = |W_i| + c_j P + d_j Q$.
  Define $|W_i|$ as, e.g., lexicographic minimum of $W_i, -W_i$.

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  Define $|W_i|$ as, e.g., lexicographic minimum of $W_i, -W_i$.
- Problem: this walk can run into fruitless cycles!
- E.g. if $|W_{i+1}| = -W_{i+1}$ and $h(|W_{i+1}|) = j = h(|W_i|)$ then

$$\begin{aligned} W_{i+2} &= f(W_{i+1}) = -W_{i+1} + c_j P + d_j Q \\ &= -(|W_i| + c_j P + d_j Q) + c_j P + d_j Q = -|W_i|, \end{aligned}$$

so $|W_{i+2}| = |W_i|$, so $W_{i+3} = W_{i+1}$, so $W_{i+4} = W_{i+2}$ etc.

Fruitless cycles

- If h maps to r different values then expect fruitless cycles of length 2 to occur with probability $1/(2r)$ at each step.
- Longer cycles occur less frequently.
- Current ECDL record, achieved 07.2009 by Bos–Kaihara–Kleinjung–Lenstra–Montgomery, "PlayStation 3 computing breaks 2^{60} barrier – 112-bit prime ECDLP solved" breaks standard curve over $\mathbb{F}_p$, where $p = (2^{128} - 3)/(11 \cdot 6949)$.
- "We did not use the common negation map since it requires branching and results in code that runs slower in a SIMD environment." (All modern CPUs are SIMD.)
- Several more papers justifying not using negation map.

Correct use of the negation map

- Our software (Bernstein-L-Schwabe, PKC 2011) solves random ECDL on the same curve (with no precomputation) in 35.6 PS3 years on average.
- For comparison, software from record computation uses 65 PS3 years on average.
- Several levels of speedups, starting with fast arithmetic mod p and continuing up through rho.
- Most important speedup: We use the negation map.
- Extra cost in each iteration:
  - extract bit of "s"(normalized y, needed anyway);
  - expand bit into mask;
  - use mask to conditionally replace (s, y) by $(-s, -y)$.
  - $\approx 1.5\%$ of total; no conditional branches.

Experiments

- To confirm iteration effectiveness we have run many experiments on $y^2 = x^3 - 3x + 9$ over the same $\mathbb{F}_p$, using smaller-order P.
- Results matched DL cost predictions.
- Conclusions of this part:
  Sensible use of negation, with or without SIMD, has negligible impact on cost of each iteration. Impact on number of iterations is almost exactly $\sqrt{2}$. Overall benefit is extremely close to $\sqrt{2}$.
- Lessons
  - Breaking the ECDLP is no harder than textbooks tell you – but that's pretty hard.
  - If a system gets broken it's not the fault of the ECDLP.

Much bigger experiments

- ECC2K-130 is the next unbroken Certicom challenge. The curve is $y^2 + xy = x^3 + 1$ considered over $\mathbb{F}_{2^{131}}$.
- For this curve we can use the negation map and the Frobenius endomorphism; this means we expect to need about $\sqrt{\pi 2^{131}/(2 \cdot 4 \cdot 2 \cdot 131)} \approx 2^{60.9}$ group operations (co-factor is 4, Frobenius has order 131).
- With a group of more than 20 collaborators have written optimized implementations for FPGAs, GPUs, the Cell, several standard CPUs – and we're running the attack.
- This will be the largest cryptanalytic computation publicly performed.
- Get more details, and watch our progress!
  http://ecc-challenge.info
  https://twitter.com/ECCchallenge

How are ECDSA systems broken?

- Anybody can produce an $R = kP$. Alice's private key is only used in $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.
- Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A. ...sometimes attacks are easier...

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- Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A. ...sometimes attacks are easier...
- If k is known for some $m, (r, s)$ then $a \equiv (sk - h(m))/r \bmod \ell$.

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- Anybody can produce an $R = kP$. Alice's private key is only used in $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.
- Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A. …sometimes attacks are easier…
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- If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r: assume $k_1 = k_2$; observe $s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$. Continue as above.

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- If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

Avoid trouble and be faster

aka "High-speed high-security signatures"
(Bernstein–Duif–L–Schwabe–Yang, CHES 2011).

- Fast signature verification. Intel Nehalem/Westmere CPUs: 280880 cycles for verification. (official eBATS measurement.)
- Even faster batch verification: < 134000 cycles/signature to verify 64 signatures of 64 messages under 64 public keys. 4×2.4GHz: 71000 verif/second!
- Very fast signing. 88328 cycles for signing. (eBATS measurement.) 4×2.4GHz: 108000 signs/second!
- Fast key generation. Almost as fast as signing.
- High security level > 2^{128} (without quantum computers).
- Small signatures and keys.

Bonus features

- No secret array indices. No information flow from secret data to addresses. $\Rightarrow$ No cache-timing attacks.
- No secret branch conditions. No information flow from secret data to branch unit.
- Collision resilience. Hash collisions do not break this signature system.
- Foolproof session keys. Signing is deterministic.
- Paper contains new representation of points which is faster for adding precomputed points.
- Curve is an Edwards curve, so it looks nicer :–)
- New efficient point compression and decompression for Edwards curves.

Some details

- Secret key of user produces a used in $P_A = aP$ and bitstring b.
- Nonce k is computed as $k =$hash(b, m).
- $R = kP$ as before but h is made dependent on R and P_A as $h =$hash(R, P_A, m).
- Signing and verification avoid inversions $s \equiv k + ha \bmod \ell$.
- Verification: $sP \stackrel{?}{=} R + hP_A$.
- Batch verification of (R_i, s_i) with $h_i =$hash(R_i, P_{A_i}, m_i) picks random scalars z_i and checks whether

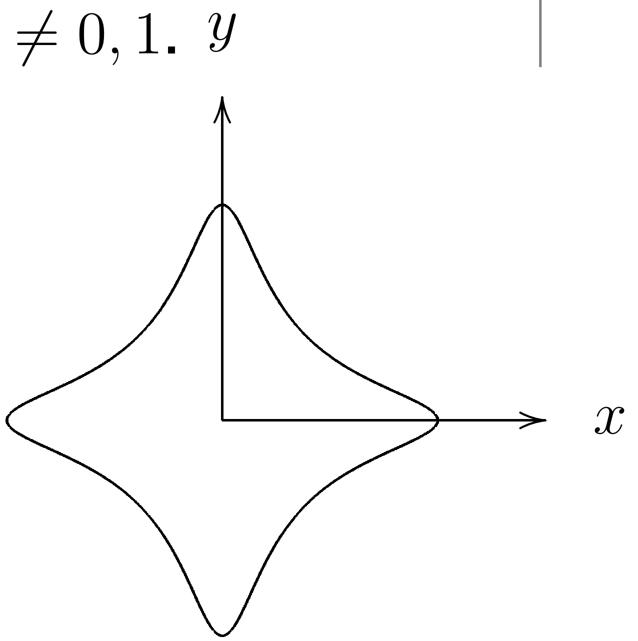
$$(-\sum_i z_i s_i \bmod \ell)P + \sum_i z_i R_i + \sum_i (z_i h_i \bmod \ell) P_{A_i} = 0.$$

Thank you for your attention!

Addition on Edwards curves

Let K be a field with $2 \neq 0$. Let $d \in K$ with $d \neq 0, 1$.
Edwards curve (nice form of elliptic curve):

$$\{(x, y) \in K \times K | x^2 + y^2 = 1 + dx^2y^2\}$$



Harold M. Edwards,
(Bulletin of the AMS, **44**, 393–422, 2007)

Associative operation on points
$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$
defined by Edwards addition law

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2} \text{ and } y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

- Neutral element is $(0, 1)$.
- $-(x_1, y_1) = (-x_1, y_1)$.

Explicit formulas: addition

- $(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right).$
- Avoid inversions: Use $(X_1 : Y_1 : Z_1)$ with $Z_1 \neq 0$ to represent $(x_1, y_1) = (X_1/Z_1, Y_1/Z_1)$, i. e., $(X_1 : Y_1 : Z_1) = (\lambda X_1 : \lambda Y_1 : \lambda Z_1)$ for $\lambda \neq 0$.
- Addition formulas in projective coordinates:

$$\begin{aligned} A &= Z_1 \cdot Z_2;\; B = A^2;\; C = X_1 \cdot X_2;\; D = Y_1 \cdot Y_2; \\ E &= d \cdot C \cdot D;\; F = B - E;\; G = B + E; \\ X_3 &= A \cdot F \cdot ((X_1 + Y_1) \cdot (X_2 + Y_2) - C - D); \\ Y_3 &= A \cdot G \cdot (D - C);\; Z_3 = F \cdot G. \end{aligned}$$

- Needs 10M + 1S + 1D + 7A, i.e. 10 multiplications, 1 squaring, 1 multiplication involving d, and 7 additions.
- Extended coordinates $(X : Y : Z : XY/Z)$

More benefits

- Doubling formulas in projective coordinates:

$$\begin{aligned} B &= (X_1+Y_1)^2;\ C = X_1^2;\ D = Y_1^2; \\ E &= C+D;\ H = Z_1^2;\ J = E-2H; \\ X_3 &= (B-E)\cdot J;\ Y_3 = E\cdot(C-D);\ Z_3 = E\cdot J. \end{aligned}$$

- Needs 3M + 4S + 6A.
- Does not benefit from extended coordinates, so output XY/Z only if next operation is addition; skip this coordinate as output of addition if followed by doubling.
- Precomputed points get stored as $(Y-X, Y+X, 2dXY)$; this corresponds to affine points $Z=1$ and saves the operations that depend only on one input point.

Beyond speed

- Neutral element of addition law is affine point.
- Addition law is symmetric in both inputs.
- $P+Q = \left(\frac{x_1y_2+y_1x_2}{1+dx_1x_2y_1y_2}, \frac{y_1y_2-x_1x_2}{1-dx_1x_2y_1y_2}\right).$
- $[2]P = \left(\frac{x_1y_1+y_1x_1}{1+dx_1x_1y_1y_1}, \frac{y_1y_1-x_1x_1}{1-dx_1x_1y_1y_1}\right).$
  Can compute this using the same formulas as general addition.
- If d is not a square in K the denominators $1+dx_1x_2y_1y_2$ and $1-dx_1x_2y_1y_2$ are never 0; addition law is complete.
- Having addition law work for doubling removes some checks from the code; addition law also works for adding $P+(-P)$ or the neutral element.