Efficient arithmetic on (hyper-)elliptic curves over finite fields

Tanja Lange

ITSC - Information-Security and Cryptography

Ruhr-University of Bochum

lange@itsc.ruhr-uni-bochum.de

12.08.2003

Overview

- Groups for DL-systems
- Arithmetic on elliptic curves
- Arithmetic on hyperelliptic curves
- Speed-up via endomorphisms
  - Koblitz curves
  - Generalized GLV
  - Trace zero subvariety

Diffie-Hellman Protocol

Alice

1. secretly generates $a < |\langle g \rangle|$
2. computes $h_1 = a \cdot g$
3. transmits h_1
4. computes $a \cdot h_2$

$= (ab)g =$

Bob

1. secretly generates $b < |\langle g \rangle|$
2. computes $h_2 = b \cdot g$
3. transmits h_2
4. computes $b \cdot h_1$

Common Key: the group element $k = (ab)g \in G$

Group G suitable

- group order can be computed efficiently,
  (avoid Pohlig-Hellman, . . .)
- discrete logarithm problem is computationally hard,
- representation is easy and compact,
- scalar multiplication is fast,
  $\Rightarrow$ key-exchange is fast.

Examples (group)

- Diffie, Hellman (1976): $G = \mathbb{F}_p^*$, i.e.

$$G = \{\, 1,\ 2,\ \ldots,\ p-1 \}$$

- $G = \mathbb{F}_q^*$, $q = p^l$
- Koblitz/Miller (1985): $G = E(\mathbb{F}_q)$, group of points on an elliptic curve
i.e.

$$G = \{\, (x, y) \in \mathbb{F}_q^2\ :\ y^2 = x^3 + ax + b \,\} \cup \{\, \infty \,\}$$

- Koblitz (1989): $G = \mathbf{Cl}(C/\mathbb{F}_q)$, ideal class group of a hyperelliptic curve C over $\mathbb{F}_q$

Arithmetic on elliptic curves

Elliptic curve

$$E : y^2 + \underbrace{(a_1x + a_3)}_{h(x)}\, y = \underbrace{x^3 + a_2x^2 + a_4x + a_6}_{f(x)},\ \ h, f \in \mathbb{F}_q[x]$$

often $q = 2^r$ or $q = p$, prime.

Group: $E(\mathbb{F}_q) = \{\, (x, y) \in \mathbb{F}_q^2 \ : \ y^2 + h(x)y = f(x) \,\} \cup \{\, \infty \,\}$

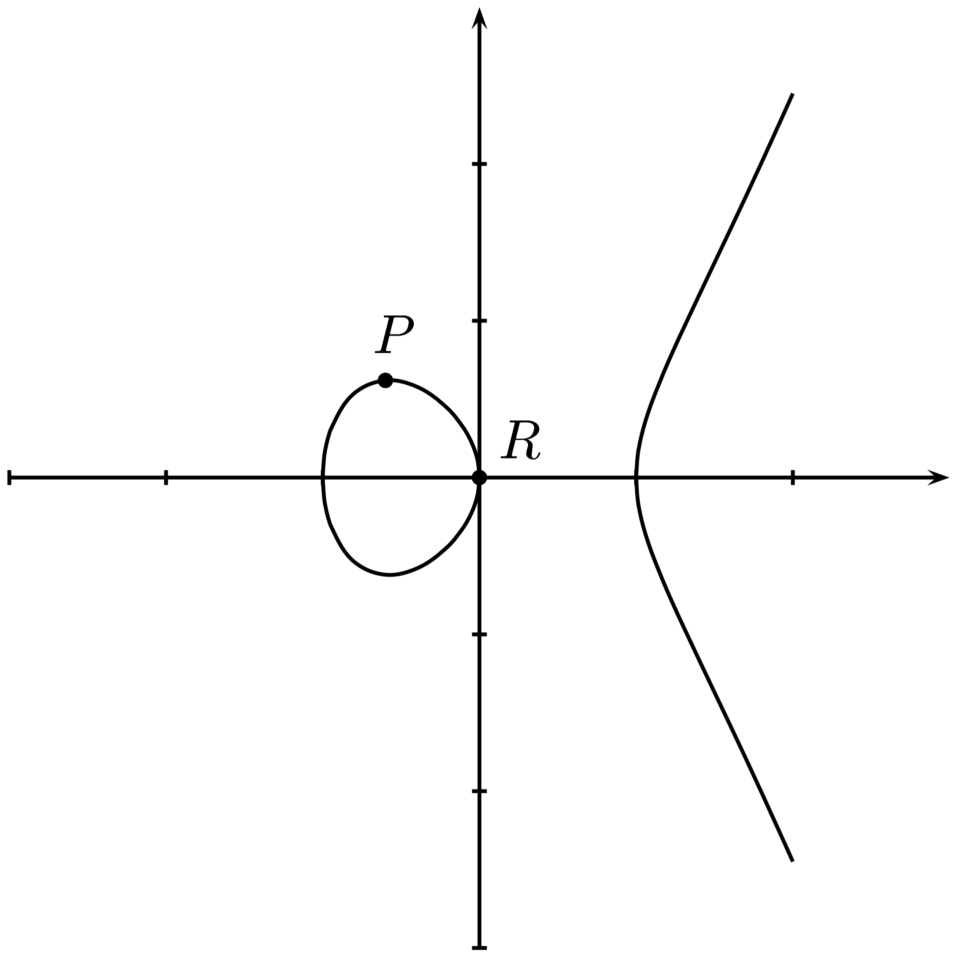
Addition law (q **odd,** $a_1 = a_2 = a_3 = 0$**)**

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3) = (\lambda^2 - x_1 - x_2, \lambda(x_1 - x_3) - y_1),$$

where $\lambda = (y_2 - y_1)/(x_2 - x_1)$ if $x_1 \neq x_2$,
$\lambda = (3x_1^2 + a_4)/(2y_1)$ else.

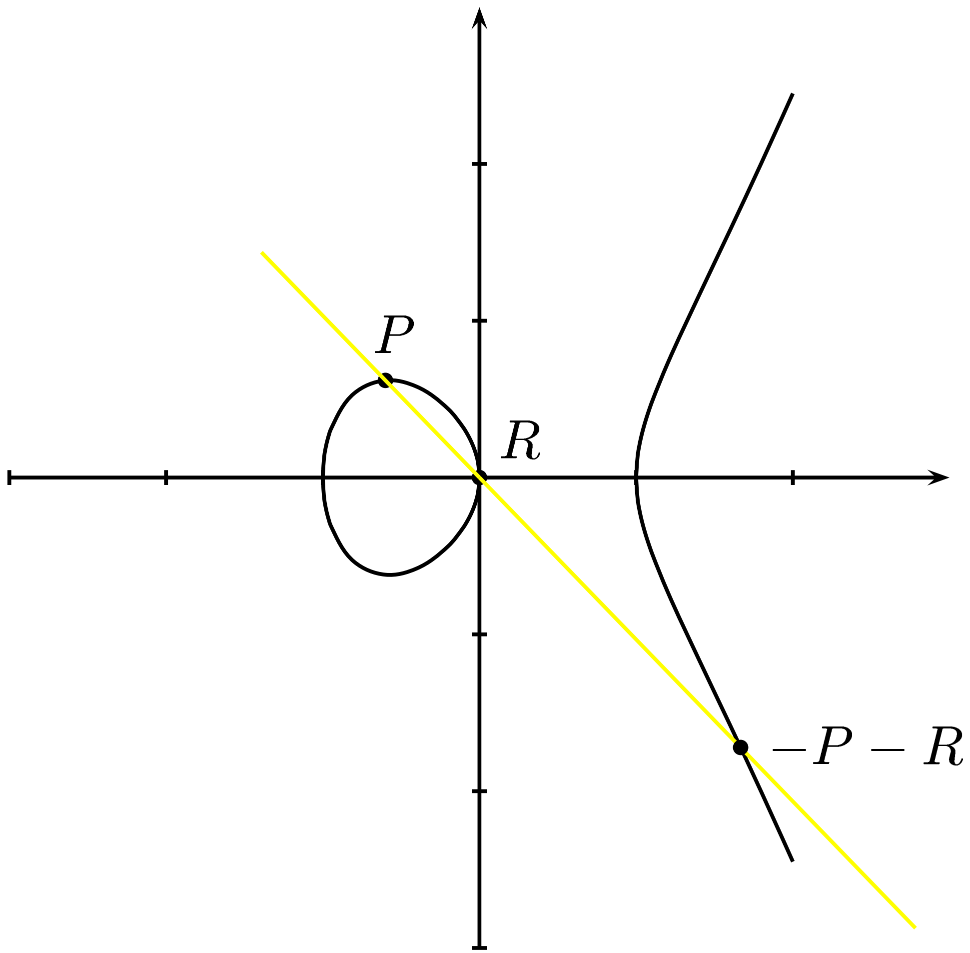
Group Law in $E(\mathbb{R})$

$y^2 = x^3 - x$



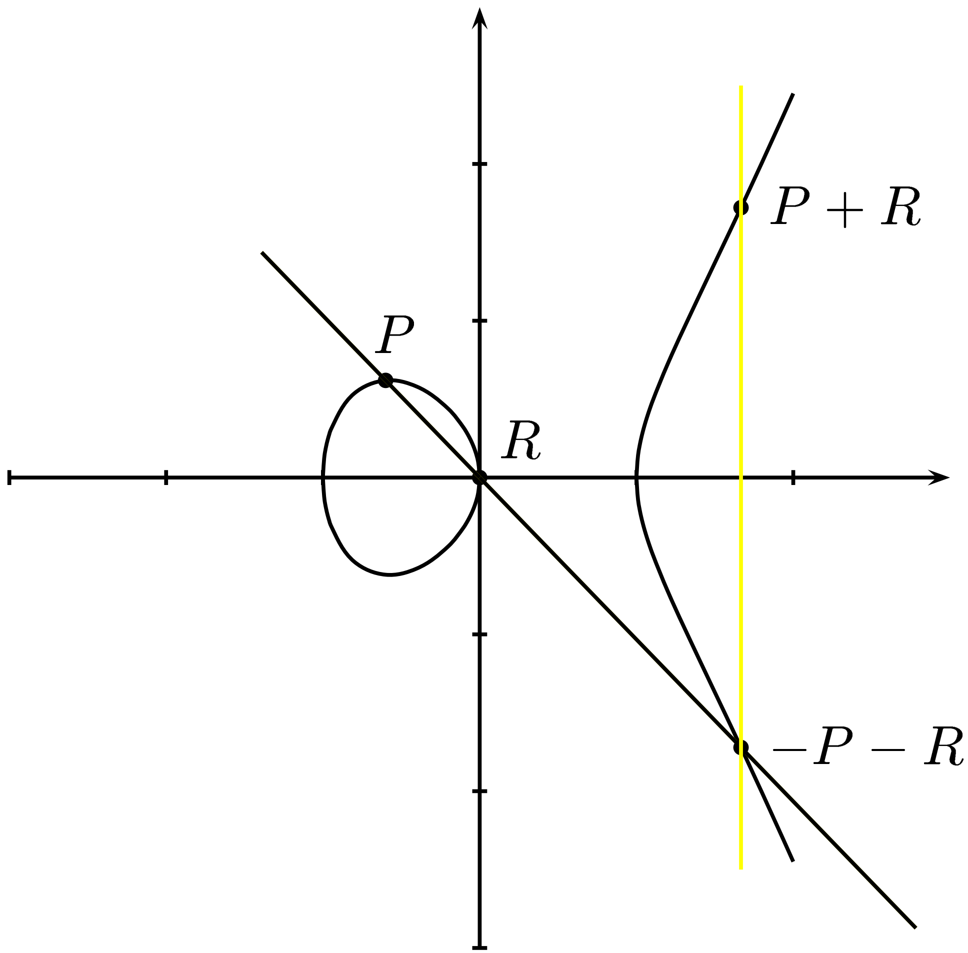
Group Law in $E(\mathbb{R})$

$y^2 = x^3 - x$



Group Law in $E(\mathbb{R})$

$y^2 = x^3 - x$



Different coordinate systems $y^2 = x^3 + ax + b$

system	points	correspondence
affine ($\mathcal{A}$)	(x, y)	
projective ($\mathcal{P}$)	(X, Y, Z)	$(X/Z, Y/Z)$
jacobian ($\mathcal{J}$)	(X, Y, Z)	$(X/Z^2, Y/Z^3)$
Chudnovsky jacobian ($\mathcal{J}^C$)	(X, Y, Z, Z^2, Z^3)	$(X/Z^2, Y/Z^3)$
modified jacobian ($\mathcal{J}^m$)	(X, Y, Z, aZ^4)	$(X/Z^2, Y/Z^3)$

system	addition			doubling		
affine ($\mathcal{A}$)	2M	1S	1I	2M	2S	1I
projective ($\mathcal{P}$)	12M	2S	–	7M	5S	–
jacobian ($\mathcal{J}$)	12M	4S	–	4M	6S	–
Chudnovsky jacobian ($\mathcal{J}^C$)	11M	3S	–	5M	6S	–
modified jacobian ($\mathcal{J}^m$)	13M	6S	–	4M	4S	–

Mixed coordinates

(Cohen, Miyaji, Ono, Asiacrypt '98)

affordable inversions:
precomputations in $\mathcal{A}$ (with Montgomery),
main doublings in $\mathcal{J}^m$,
final doublings $2\mathcal{J}^m = \mathcal{J}$,
additions $\mathcal{A} + \mathcal{J} = \mathcal{J}^m$

expensive inversions:
precomputations in $\mathcal{J}^C$,
main doublings in $\mathcal{J}^m$,
final doublings $2\mathcal{J}^m = \mathcal{J}$,

additions $\mathcal{J} + \mathcal{J}^C = \mathcal{J}^m$

Arithmetic on hyperelliptic curves

Hyperelliptic curve

$$C : y^2 + h(x)y = f(x)$$

$h(x), f(x) \in \mathbb{F}_q[x],\ q$ prime power,
f monic, $\deg f = 2g + 1, \deg h \leq g$
nonsingular, i. e. not both partial derivatives zero for $(a, b) \in C/\bar{\mathbb{F}}_q$

C **hyperelliptic curve of genus** g

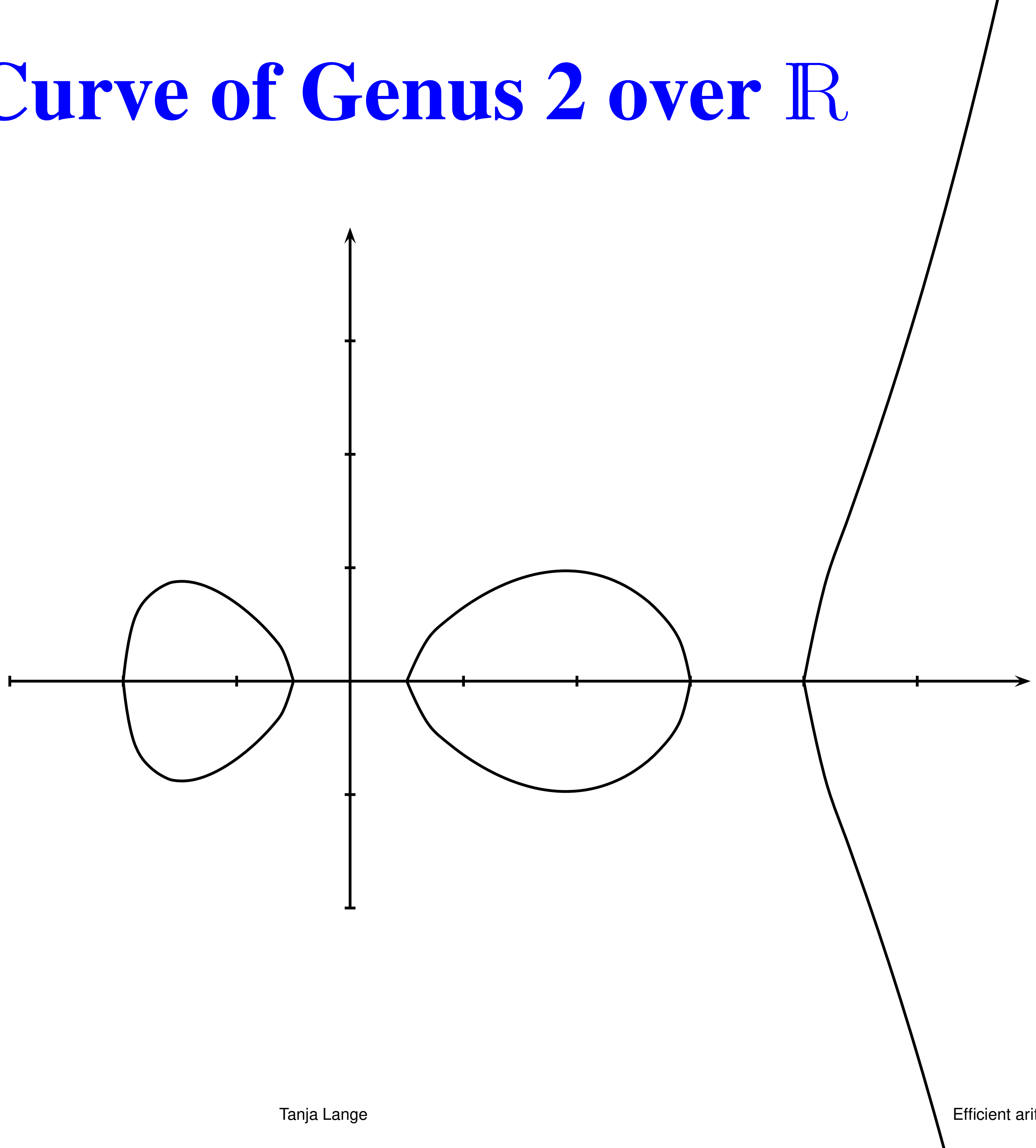
with at least one $\mathbb{F}_q$-rational Weierstraß point
(elliptic curves are curves of genus 1)

Example:

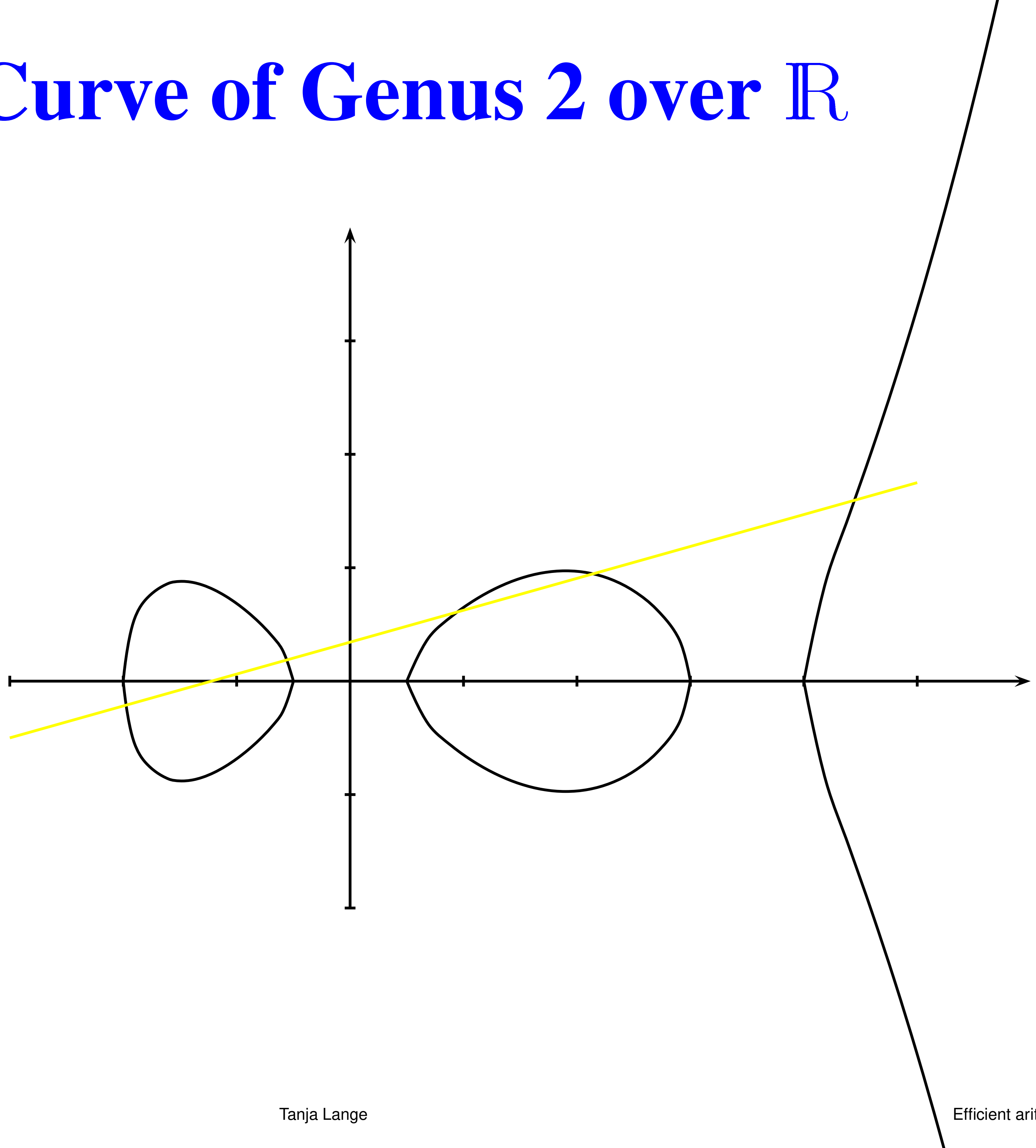
$$y^2 = x^5 + f_3x^3 + f_2x^2 + f_1x + f_0$$

genus 2 curve over field of odd characteristic.

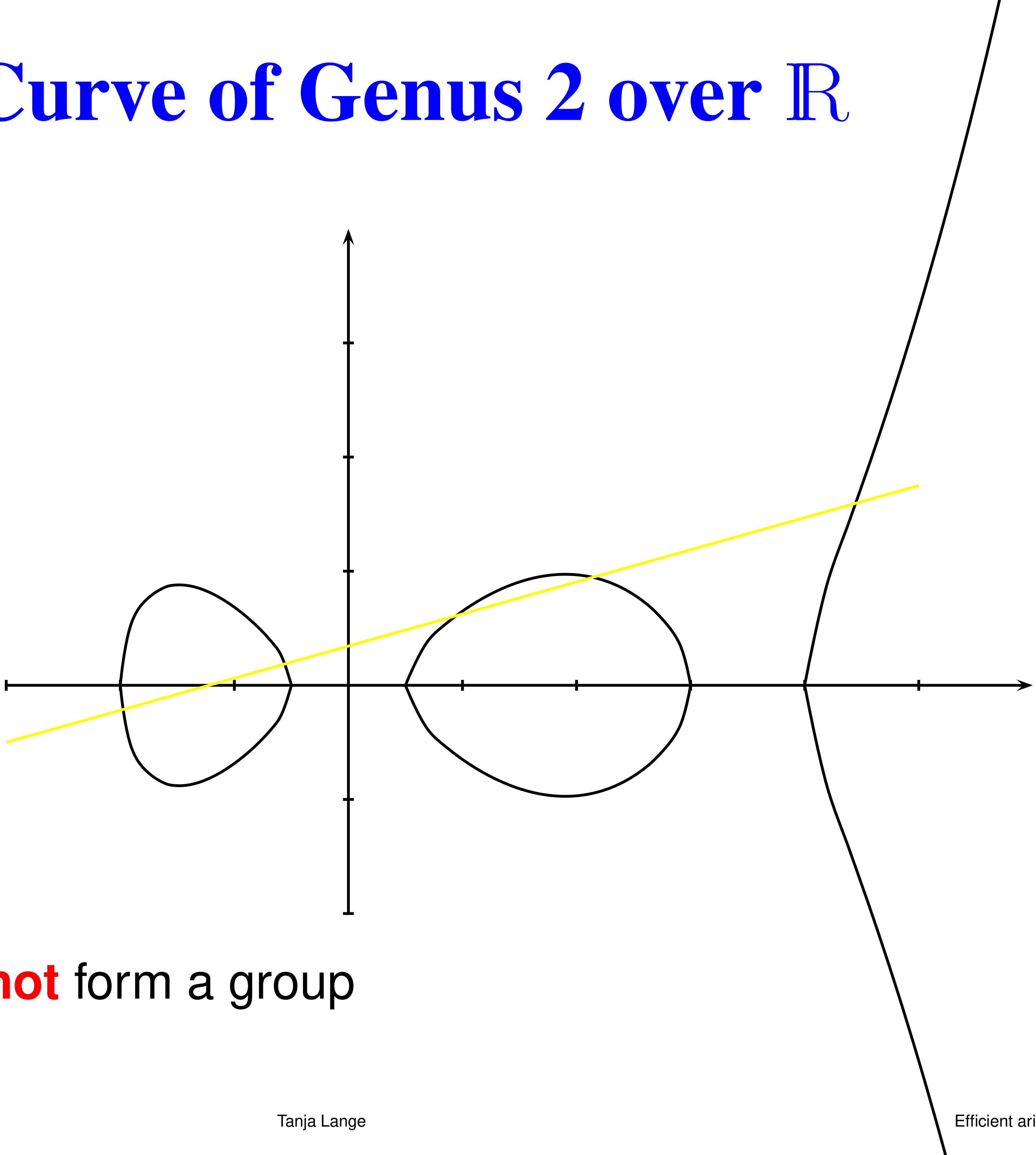
Curve of Genus 2 over $\mathbb{R}$



Curve of Genus 2 over $\mathbb{R}$



Curve of Genus 2 over $\mathbb{R}$



Points do **not** form a group

Group

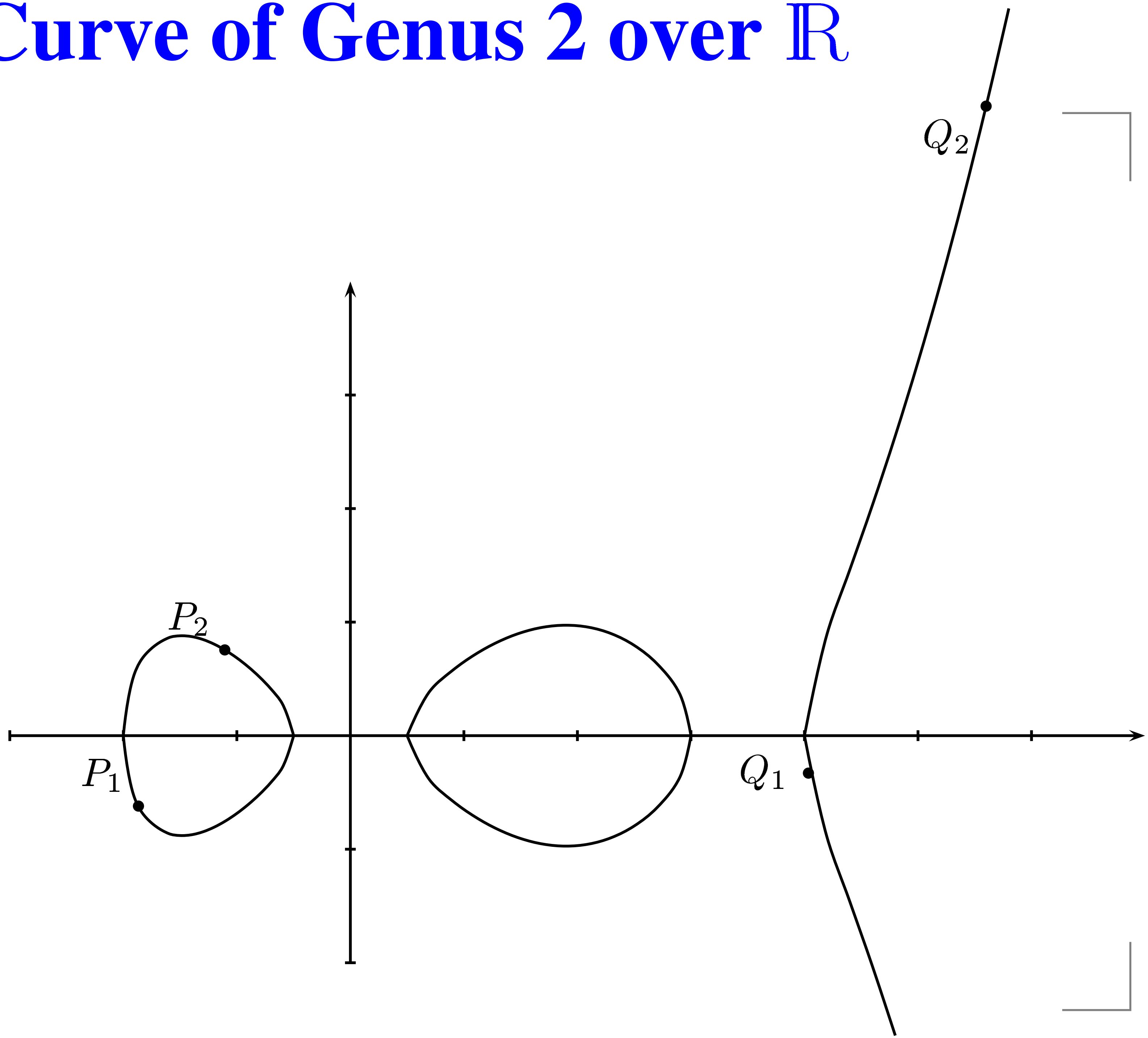
Use **divisors**, i. e. sums of points, of degree zero and reduce modulo principal divisors
$\Rightarrow$ **divisor class group**.

$$\mathrm{Pic}^0(C/\mathbb{F}_q) = Div^0(\mathbb{F}_q)/P(\mathbb{F}_q)$$

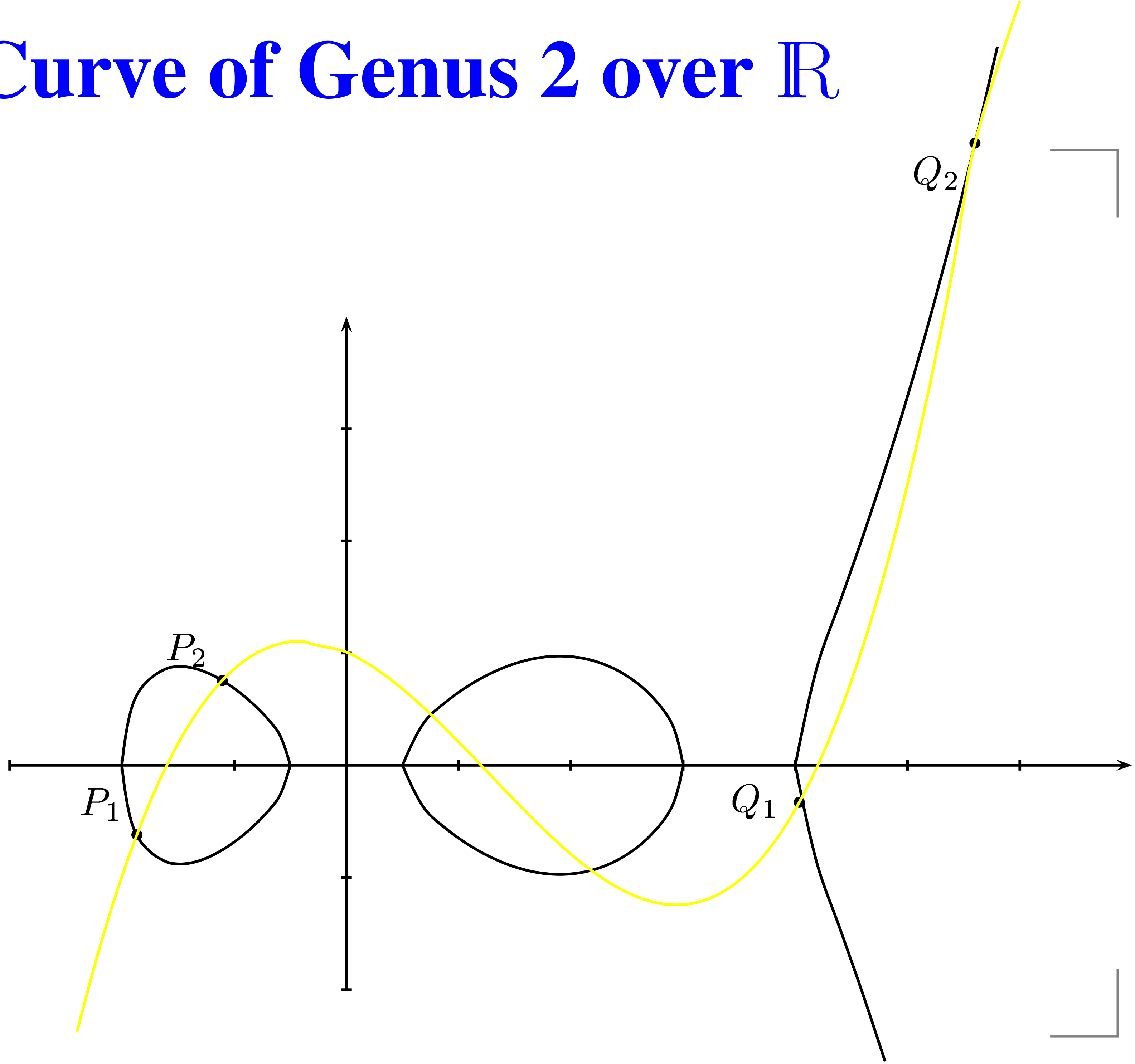
each class represented by degree zero divisor with $m \leq g$

$$\sum_{\substack{i=1 \\ P_i \in C\setminus\{\infty\}}}^{m} P_i - m\infty$$

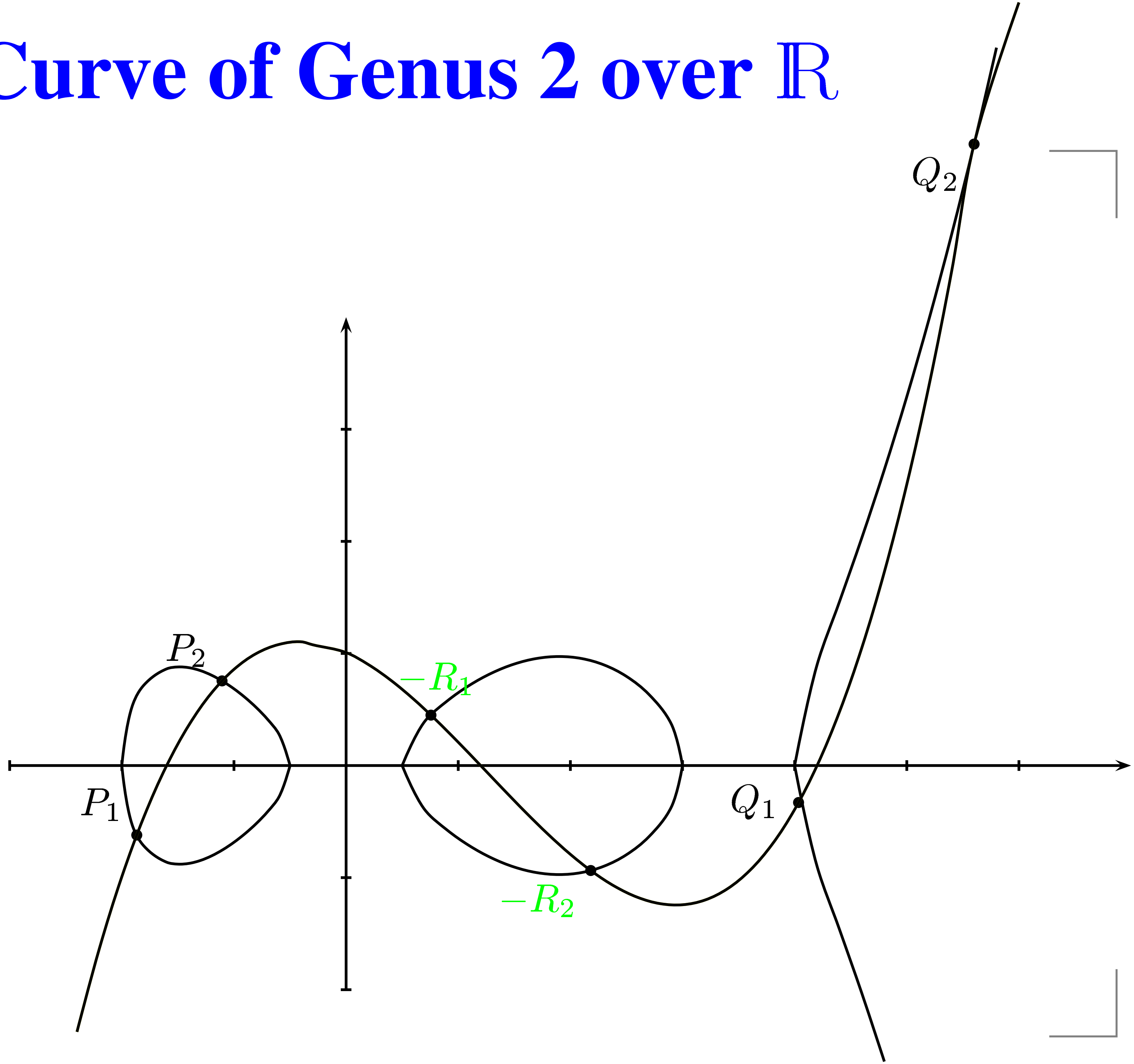
Curve of Genus 2 over $\mathbb{R}$



Curve of Genus 2 over $\mathbb{R}$

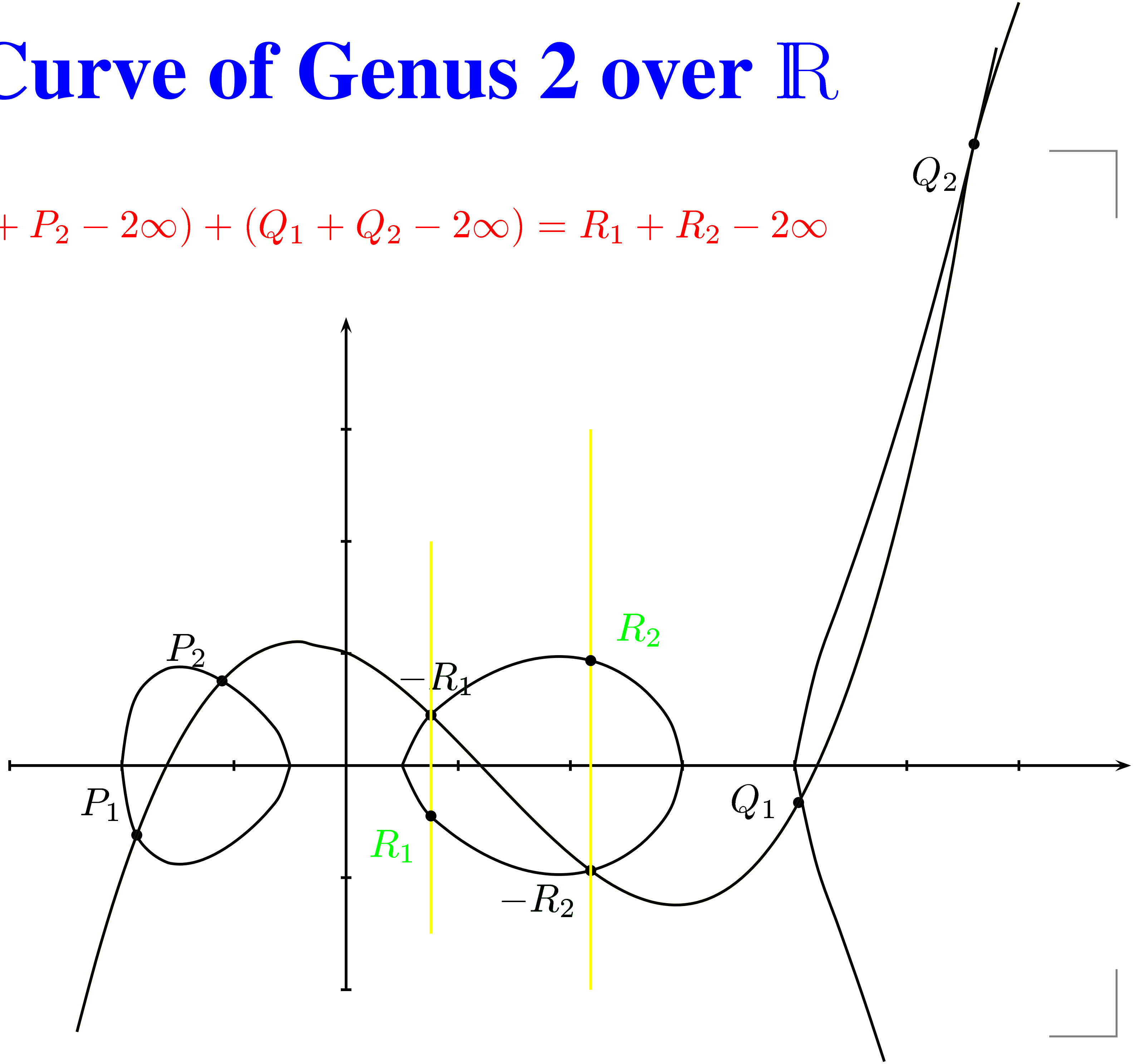


Curve of Genus 2 over $\mathbb{R}$



Curve of Genus 2 over $\mathbb{R}$

$$(P_1 + P_2 - 2\infty) + (Q_1 + Q_2 - 2\infty) = R_1 + R_2 - 2\infty$$



Mumford Representation

Divisor class group is isomorphic to
ideal class group $\mathrm{Cl}(C/\mathbb{F}_q)$ of $\mathbb{F}_q(x,y)/(C)$.
Elements D represented by two polynomials

$$D = [u(x), v(x)]; u, v \in \mathbb{F}_q[x],$$

u monic, $\deg v < \deg u \leq g, \quad u | v^2 + vh - f.$

$\Rightarrow$ compact representation

isomorphism via

$$P_i = (x_i, y_i) \Leftrightarrow u(x_i) = 0, v(x_i) = y_i \text{ with multiplicity}$$

Arithmetic on hyperelliptic curves

Composition & Reduction (Cantor/Koblitz)

`IN:` $D_1 = [u_1, v_1], D_2 = [u_2, v_2], C : y^2 + h(x)y = f(x)$
`OUT:` $D = [u, v]$ reduced with $D \sim D_1 + D_2$

1. compute $d_1 = \gcd(u_1, u_2) = e_1 u_1 + e_2 u_2$
2. compute $d = \gcd(d_1, v_1 + v_2 + h) = c_1 d_1 + c_2(v_1 + v_2 + h)$
3. let $s_1 = c_1 e_1, s_2 = c_1 e_2, s_3 = c_2$
4. $u = \frac{u_1 u_2}{d^2} \quad v = \frac{s_1 u_1 v_2 + s_2 u_2 v_1 + s_3(v_1 v_2 + f)}{d} \bmod u$
5. let $u' = \frac{f - vh - v^2}{u} \quad b' = (-h - v) \bmod u'$
6. if $\deg u' > g$ put $u := u', v := v'$ goto step 5
7. make u monic

Arithmetic in $\mathbf{Cl}(C/\mathbb{F}_q)$

$$\left.\begin{array}{ll} 2000 & \text{Harley (odd char.)} \\ 2001 & \text{L. (arbitrary char.)} \\ 2001 & \text{Matsuo, Chao, Tsujii (faster)} \end{array}\right\} \Rightarrow \text{2 inversions}$$

$$\left.\begin{array}{ll} 2002 & \text{Miyamoto, Doi, Matsuo, Chao, Tsujii} \\ 2002 & \text{Takahashi} \\ 2002 & \text{L. (arbitrary char.)} \\ 2002 & \text{Sugizaki, Matsuo, Chao, Tsujii} \\ & \text{(even char.)} \end{array}\right\} \Rightarrow \text{1 inv.}$$

genus 3

$$\left.\begin{array}{ll} 2002 & \text{Kuroki, Gonda, Matsuo, Chao, Tsujii} \\ 2002 & \text{Pelzl, Guyot \& Patankar} \end{array}\right\} \Rightarrow \text{1 inv.}$$

Addition, $g = 2$

Addition, $\deg u_1 = \deg u_2 = 2$		
Input	$[u_1, v_1], [u_2, v_2], u_i = x^2 + u_{i1}x + u_{i0}, v_i = v_{i1}x + v_{i0}$	
Output	$[u', v'] = [u_1, v_1] + [u_2, v_2]$	
Step	Expression	Operations
1	compute resultant r of u_1, u_2: $z_1 = u_{11} - u_{21}, z_2 = u_{20} - u_{10}, z_3 = u_{11}z_1 + z_2;$ $r = z_2z_3 + z_1^2u_{10};$	1S, 3M
2	compute almost inverse of u_2 modulo u_1 ($inv = r/u_2 \bmod u_1$): $inv_1 = z_1, inv_0 = z_3;$	
3	compute $s' = rs \equiv (v_1 - v_2)inv \bmod u_1$: $w_0 = v_{10} - v_{20}, w_1 = v_{11} - v_{21}, w_2 = inv_0w_0, w_3 = inv_1w_1;$ $s'_1 = (inv_0 + inv_1)(w_0 + w_1) - w_2 - w_3(1 + u_{11}), s'_0 = w_2 - u_{10}w_3;$ if $s'_1 = 0$ see below	5M
4	compute $s'' = x + s_0/s_1 = x + s'_0/s'_1$ and s_1: $w_1 = (rs'_1)^{-1}(= 1/r^2s_1), w_2 = rw_1(= 1/s'_1), w_3 = {s'}_1^2w_1(= s_1);$ $w_4 = rw_2(= 1/s_1), w_5 = w_4^2, s''_0 = s'_0w_2;$	I, 2S, 5M
5	compute $l' = s''u_2 = x^3 + l'_2x^2 + l'_1x + l'_0$: $l'_2 = u_{21} + s''_0, l'_1 = u_{21}s''_0 + u_{20}, l'_0 = u_{20}s''_0$	2M
6	compute $u' = (s(l + h + 2v_2) - k)/u_1 = x^2 + u'_1x + u'_0$: $u'_0 = (s''_0 - u_{11})(s''_0 - z_1 + h_2w_4) - u_{10} + l'_1 + (h_1 + 2v_{21})w_4 + (2u_{21} + z_1 - f_4)w_5;$ $u'_1 = 2s''_0 - z_1 + h_2w_4 - w_5;$	3M
7	compute $v' \equiv -h - (l + v_2) \bmod u' = v'_1x + v'_0$: $w_1 = l'_2 - u'_1, w_2 = u'_1w_1 + u'_0 - l'_1, v'_1 = w_2w_3 - v_{21} - h_1 + h_2u'_1;$ $w_2 = u'_0w_1 - l'_0, v'_0 = w_2w_3 - v_{20} - h_0 + h_2u'_0;$	4M
total		I, 3S, 22M

Different Coordinates $g = 2$

affine $\mathcal{A}$:

$$D \sim [x^2 + u_1 x + u_0, v_1 x + v_0] \sim [u_1, u_0, v_1, v_0]$$

projective $\mathcal{P}$: Miyamoto, Doi, Matsuo, Chao, and Tsujii; L.; all 2002

$$D \sim [U_1, U_0, V_1, V_0, Z] \sim [U_1/Z, U_0/Z, V_1/Z, V_0/Z]$$

new $\mathcal{N}$: L. 2002, contains also mixed coordinates

$$D \sim [U_1, U_1, V_1, V_0, Z_1, Z_2] \sim [U_1/Z_1^2, U_0/Z_1^2, V_1/(Z_1^3 Z_2), V_0/(Z_1^3 Z_2)]$$

$\mathcal{P}$ and $\mathcal{N}$: **no inversions**

Comparison - odd characteristic

Doubling		Addition	
operation	costs	operation	costs
$2\mathcal{N} = \mathcal{P}$	7S, 38M	$\mathcal{N} + \mathcal{N} = \mathcal{P}$	7S, 51M
$2\mathcal{P} = \mathcal{P}$	6S, 38M	$\mathcal{N} + \mathcal{P} = \mathcal{P}$	4S, 51M
$2\mathcal{N} = \mathcal{N}$	7S, 34M	$\mathcal{N} + \mathcal{N} = \mathcal{N}$	7S, 47M
$2\mathcal{P} = \mathcal{N}$	6S, 34M	$\mathcal{N} + \mathcal{P} = \mathcal{N}$	4S, 48M
		$\mathcal{P} + \mathcal{P} = \mathcal{P}$	4S, 47M
		$\mathcal{P} + \mathcal{P} = \mathcal{N}$	4S, 44M
		$\mathcal{A} + \mathcal{N} = \mathcal{P}$	5S, 40M
		$\mathcal{A} + \mathcal{P} = \mathcal{P}$	3S, 40M
		$\mathcal{A} + \mathcal{N} = \mathcal{N}$	5S, 36M
		$\mathcal{A} + \mathcal{P} = \mathcal{N}$	3S, 37M
$2\mathcal{A} = \mathcal{A}$	1I, 5S, 22M	$\mathcal{A} + \mathcal{A} = \mathcal{A}$	1I, 3S, 22M

Implementations and Comparison

- **software:** for $\log_2 p \sim 40 - 60$ inversions cheap,
  use $\mathcal{A}$ for $g = 1, 2$.
  Comparison depends on I/M genus
  Upcoming paper by Roberto Avanzi (Essen), library for small p.
- **hardware:** usually expensive inversions,
  compute multiple of affine point via
  $2\mathcal{N} = \mathcal{N}, \mathcal{A} + \mathcal{N} = \mathcal{N}$,
  compute multiple of non-normalized point via
  $2\mathcal{N} = \mathcal{N}$, $\mathcal{N} + \mathcal{P} = \mathcal{N}$.
  - Implementation by Kim Nguyen (Philips) presented at ECC 2002 using $\mathcal{P}$ is half as fast as elliptic
  - talk by Christof Paar (Bochum)

Speed-up via endomorphisms

Koblitz curves

Koblitz curves (subfield curves)

$$C : y^2 + h(x)y = f(x)$$

hyperelliptic curve with
$h(x), f(x) \in \mathbb{F}_q[x]$, q **small**
consider curve over $\mathbb{F}_{q^n}$

Example:
$C : y^2 + xy = x^5 + x^2 + 1$
hyperelliptic curve of genus 2 over $\mathbb{F}_2$
hyperelliptic curve of genus 2 over $\mathbb{F}_{2^n}$

History: Koblitz ’89; Günther, L., Stein ’00; L. 01 (Ph.D.)

Frobenius endomorphism σ :

$$P = (a, b) \in C(\mathbb{F}_{q^n})$$

$$\Rightarrow \sigma(P) := (a^q, b^q) \in C(\mathbb{F}_{q^n})$$

Normal basis

Basis of $\mathbb{F}_{q^n} = \{\theta, \theta^q, \theta^{q^2}, \ldots, \theta^{q^{n-1}}\}$
Element $d \in \mathbb{F}_{q^n}$ represented by

$$d = (d_0, d_1, \ldots, d_{n-1}), \quad d_i \in \mathbb{F}_q$$

$$d^q = (d_{n-1}, d_0, d_1, \ldots, d_{n-2})$$

Remark: d^q realized by **cyclic shifting**,

q-th power also fast in polynomial basis

Frobenius endomorphism on $\mathrm{Cl}(C/\mathbb{F}_q)$

Operation on ideal classes $D = [u(x), v(x)]$

$$\sigma(D) = [\sigma(u(x)), \sigma(v(x))],$$

where $\sigma(\sum d_i x^i) = \sum d_i^q x^i$.
Execution of Frobenius endomorphism takes at most $2g$ cyclic shiftings in normal basis representation
(L., Nöcker: polynomial basis still faster)

Example:
C curve of genus 2 defined over $\mathbb{F}_2$

$$D = [x^2 + u_1 x + u_0, v_1 x + v_0] \quad \sigma(D) = [x^2 + u_1^2 x + u_0^2, v_1^2 x + v_0^2]$$

Characteristic polynomial of σ

$$P(T) = T^{2g} + a_1 T^{2g-1} + \cdots + a_g T^g + \cdots + a_1 q^{g-1} T + q^g$$

integer coefficients, degree $2g$

Examples for genus 2 over $\mathbb{F}_2$

Equation of C	$P(T)$
$v^2 + v = u^5 + u^3$	$T^4 + 2T^3 + 2T^2 + 4T + 4$
$v^2 + v = u^5 + u^3 + 1$	$T^4 - 2T^3 + 2T^2 - 4T + 4$
$v^2 + v = u^5 + u^3 + u$	$T^4 + 2T^2 + 4$
$v^2 + uv = u^5 + 1$	$T^4 + T^3 + 2T + 4$
$v^2 + uv = u^5 + u^2 + 1$	$T^4 - T^3 - 2T + 4$
$v^2 + (u^2 + u + 1)v = u^5 + u^4 + u^3$	$T^4 + T^2 + 4$
$v^2 + (u^2 + u)v = u^5 + u^4 + u$	$T^4 - T^2 + 4$
$v^2 + (u^2 + u + 1)v = u^5 + u^4$	$T^4 + 2T^3 + 3T^2 + 4T + 4$
$v^2 + (u^2 + u + 1)v = u^5 + u^4 + 1$	$T^4 - 2T^3 + 3T^2 - 4T + 4$

$P(T)$ **easy** to compute for **Koblitz curves**

Consequences I

$$-q^g D = \sigma^{2g}(D) + a_1\sigma^{2g-1}(D) + \cdots + a_1 q^{g-1}\sigma(D)$$

for $D \in \mathbf{Cl}(C/\mathbb{F}_q)$.

$c = c_1 + c_2\tau + \cdots + c_{2g}\tau^{2g-1} \in \mathbb{Z}[\tau]$

$$\tau | c \Leftrightarrow q^g | c_1$$

Necessary for expansion: set of coefficients

$$R \supseteq \{0, \pm 1, \ldots, \pm\lceil \frac{q^g-1}{2} \rceil\}$$

Note:

Taking negative is for free.

$$D = [u(x), v(x)] \Rightarrow -D = [u(x), -h(x) - v(x)]$$

Consequences II

To compute $mD = \sum u_i \sigma^i(D)$ use τ**-adic expansion**

$$m = \sum u_i \tau^i,$$

where τ complex root of $P(T)$, $u_i \in R$: **set of coefficients.**

Results:

- if finite, expansions have length $\sim n$
- if not finite, then periodic with short period
- periodic expansions can easily be avoided

Comparison

Operations in **binary** representation:

$$\sim \frac{3}{2}\, g\, n\, \log_2 q \geq \frac{3}{2}\, g\, n$$

Operations in τ**-adic** representation:

- $(q^g - 2)/2$ precomputations

$$\sim \frac{q^g - 1}{q^g}\, n < n$$

- $(q^g - 1)q^g/2$ precomputations

$$\sim \frac{q^g - 1}{2q^g - 1}\, n < \frac{1}{2}\, n$$

Comparison

$q = 2$ same number of precomputations as signed binary

g	binary window	τ-adic $w=1$	speed-up	binary window	τ-adic $w=2$	speed-up
2	$11/4n$	$3/4n$	11/3	$31/12n$	$3/7n$	~ 6
3	$31/8n$	$7/8n$	31/7	$573/160n$	$7/15n$	~ 7.6
4	$79/16n$	$15/16n$	79/15	$1023/224n$	$15/31n$	~ 9.4

$q = 5$ comparison friendly towards ordinary curves

g	w_{bin}	binary window	τ-adic $w=1$	speed-up	w_{bin}	binary window	τ-adic $w=2$	speed-up
2	4	$47/8n$	$24/25n$	~ 6	9	$6n$	$24/49n$	~ 12
3	7	$511/64n$	$\sim n$	~ 8	13	$8n$	$124/249n$	~ 16
4	9	$10n$	$\sim n$	~ 10	18	$10n$	$\sim 1/2n$	~ 20

Generalized GLV

Other endomorphisms – GLV

- Gallant, Lambert, and Vanstone
- Park, Jeong, Kim, and Lim
- Sica, Ciet, and Quisquater

ϕ endomorphism of C, $P_\phi(T)$ characteristic polynomial of ϕ, $d = \deg P_\phi \leq 2g$, l group order.
split

$$m = m_0 + m_1\phi + \cdots + m_{d-1}\phi^{d-1}, |m_i| \sim l^{1/d}$$

for $d = 2$ use joint-sparse form (JSF) of exponent, otherwise interleaved multiplications.

Both significantly faster than binary method

Combination of GLV and expansion

Ciet, L., Sica, Quisquater, Eurocrypt '03

ϕ satisfies

$$T^2 + rT + s$$

use GLV

$$m = m_0 + m_1\phi, \quad |m_i| \sim l^{1/2}$$

expand

$$m_0 + m_1\phi = k_0 + k_1\phi + k_2\phi^2 + \cdots, \; k_i \in \mathcal{R}, \text{length} \sim \log_s m$$

precompute $r_i P$ for $r_i \in \mathcal{R} = \{0, \pm 1, \ldots, \lceil (s-1)/2 \rceil\}$

together with ϕ-JSF $\Rightarrow$ further speed-up if computation of $\phi(P)$ less expansive than s-fold

Trace zero variety

Trace-zero subvariety

genus 1 or 2 curve defined over $\mathbb{F}_p$, considered over $\mathbb{F}_{p^n}$,
where $n \in \{3,5\}$ for $g=1$ and $n=3$ for $g=2$
trace zero property

$D \in G \Leftrightarrow D \in \mathbf{Cl}(C/\mathbb{F}_{p^n})$ and $D + \sigma(D) + \cdots + \sigma^{n-1}(D) = 0,$

subgroup of $\mathbf{Cl}(C/\mathbb{F}_{p^n})$
('Koblitz curves' for small extension)

History: Frey '98, Naumann '99 ($g=1$),
L. '01 ($g=2$), Blady '02 ($g=1$)
Silverberg/Rubin '02 (supersingular curves)

Avanzi, Diem, Frey, L., Scholten '03

Background

dimension $(n-1)g$, group size from $P(T)$,

$$\#G \sim \#\mathbf{Cl}(C/\mathbb{F}_{p^n})/\#\mathbf{Cl}(C/\mathbb{F}_p) \sim p^{(n-1)g}$$

Theorem *Put* $g=2, n=3$*. Let* $2,3 \not| l \,\big|\, \#\mathbf{Cl}(C/\mathbb{F}_{p^3})$*. The nontrivial elements in the subgroup of order* l *of the trace zero variety* G *are the divisor classes represented by*

$$P_1 + P_2 - 2\infty \notin \mathrm{Div}(C/\mathbb{F}_p)^0,$$

where $P_1 \neq P_2, \sigma(P_2), \sigma^2(P_2)$*.*

Scalar multiples in G

Frobenius endomorphism satisfies characteristic polynomial

$P(T) = T^2 + a_1 T + p$ resp. $P(T) = T^4 + a_1 T^3 + a_2 T^2 + a_1 p T + p^2$.

and trace relation

$$T^2 + T + 1 \text{ resp. } T^4 + T^3 + T^2 + T + 1$$

instead of computing mD for $0 \leq m < l$, $l \sim p^{(n-1)g}$ group oder $\langle D \rangle \subseteq G$ use

$$r_0 D + \cdots r_{n-2} \sigma(D), \ |r_i| \sim \sqrt[n]{m}.$$

and (pairwise) Joint Sparse Form (JSF) of r_i.

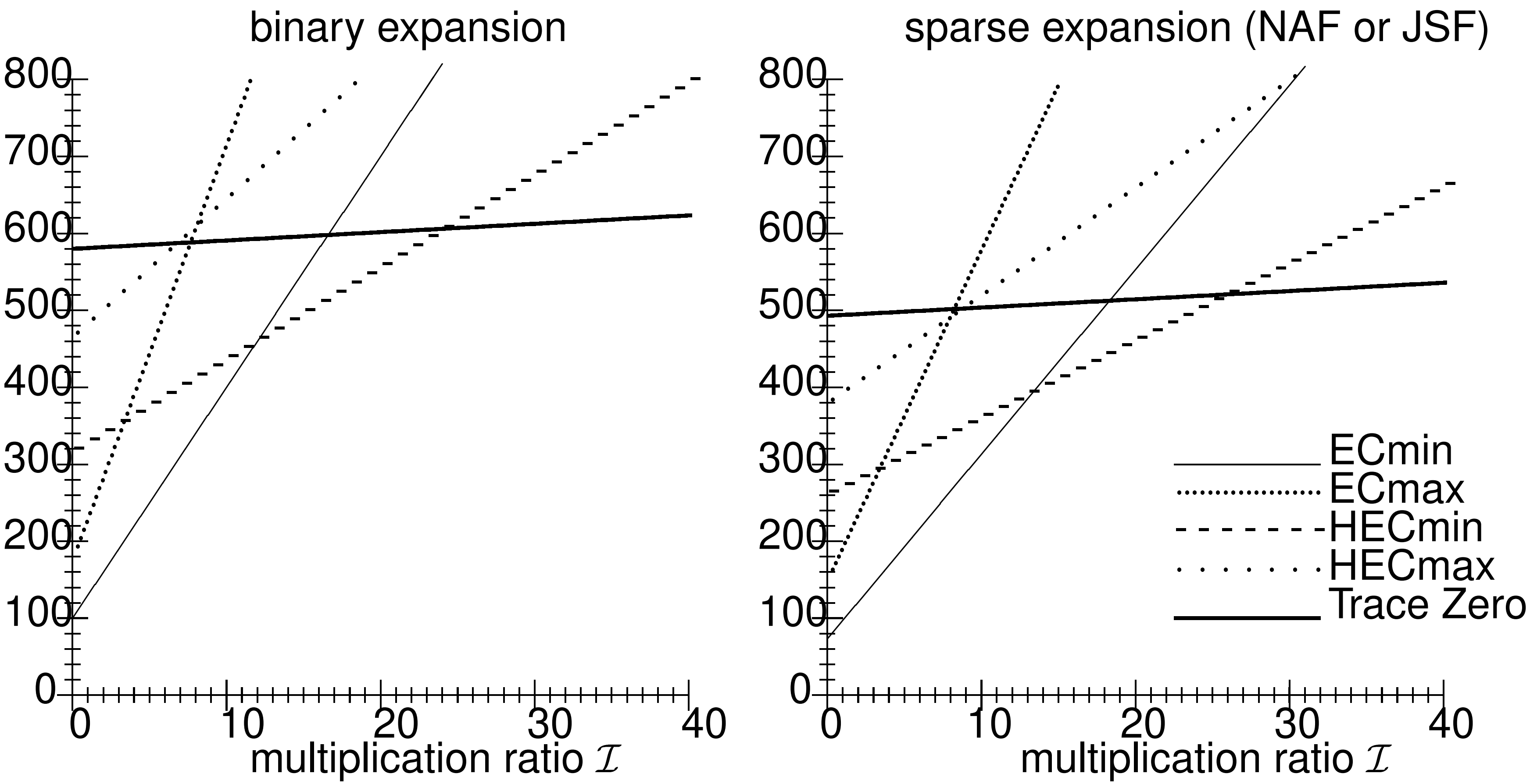
Comparison I, $g = 2, n = 3$

	Elliptic, in $\mathbb{F}_{p''}$			Genus 2, in $\mathbb{F}_{p'}$			G ($g = 2$), in $\mathbb{F}_p$		
	Inv.	Sqr.	Mult.	Inv.	Sqr.	Mult.	Inv.	Sqr.	Mult.
Add.	1	1	2	1	3	22	1	21	141
Doub.	1	2	2	1	5	22	1	33	141
m-fold	6λ	10λ	12λ	6λ	26λ	132λ	3.5λ	97.5λ	501.5λ
NAF/ JSF	4.8λ	8.8λ	9.6λ	4.8λ	22.4λ	105.6λ	3λ	87λ	423λ

using efficient arithmetic in $\mathbb{F}_{p^3}$

(NAF includes same number of precomputations)

Comparison II, $g = 2, n = 3$



Implementation: work in progress by Avanzi, L.

The End

The END

Thanks!