Pairings on elliptic and hyperelliptic curves

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Overview

- Protocols and hardness assumptions
- Divisor class group
- Tate and Weil pairing on elliptic curves
- Efficient computation of pairings
- Supersingular and ordinary elliptic curves
- Distortion maps
- Embedding degree
- MNT curves
- Outlook

Protocols and hardness assumptions

DL systems

These systems assume that the Discrete Logarithm Problem (DLP) is hard to solve, i.e.

- given P and $P_A = [s_A]P$
- it is hard to find s_A.

The Computational Diffie-Hellman Problem (CDHP) is the problem

- given P, $P_A = [s_A]P$, and $P_B = [s_B]P$
- compute $[s_A s_B]P$.

The Decisional Diffie-Hellman Problem (DDHP) is the problem

- given P, $P_A = [s_A]P$, $P_B = [s_B]P$ and $R = [r]P$
- decide whether $R = [s_A s_B]P$.

Pairings

Let $(G_1, +), (G'_1, +)$ and $(G, \cdot)$ be groups and let

$$e : G_1 \times G'_1 \to G$$

be a map satisfying

- $e(P + Q, R') = e(P, R')e(Q, R')$
- $e(P, R' + S') = e(P, R')e(P, S')$
- The map is non-degenerate in the first argument, i.e. if $e(P, R') = 1$ for all $R' \in G'_1$ for some P then P is the identity in G_1

Then e is called a bilinear map or pairing.

In protocol papers often $G_1 = G'_1$.

Consequences

- Assume that $G_1 = G_1'$ and hence
$$e(P, P) \neq 1.$$
Then for all triples $(P_1, P_2, P_3) \in \langle P \rangle^3$ one can decide in time polynomial in $\log \ell$ whether
$$\log_P(P_3) = \log_P(P_1) \log_P(P_2)$$
by comparing
$$e(P_1, P_2) \stackrel{?}{=} e(P, P_3).$$
Thus the DDHP is easy.
- The DL system G_1 is at most as secure as the system G. Even if $G_1 \neq G_1'$ one can transfer the DLP in G_1 to a DLP in G, provided that one can find an element $P' \in G_1'$ such that the map $P \to e(P, P')$ is injective.

Pairing based protocols I

Joux, ANTS 2000, one round tripartite key exchange

Let P, P' be generators of G_1 and G_1' respectively.
Users A, B and C compute joint secret from their secret contributions a, b, c as follows (A's perspective)

- Compute and send $[a]P, [a]P'$.
- Upon receipt of $[b]P$ and $[c]P'$ put $k = (e([b]P, [c]P'))^a$

The resulting element k is the same for each participant as

$$k = (e([b]P, [c]P'))^a = (e(P, P'))^{abc} = (e([a]P, [c]P'))^b = (e([a]P, [b]P'))^c.$$

- Obvious saving in first step if $G_1 = G_1'$.
- Only one user needs to do both computations.

Pairing based protocols II

- Sakai-Ohgishi-Kasahara, 2000 ID-based key exchange
- Boneh and Franklin, Crypto 2001, ID-based cryptography

Idea: user's identity defines his public key.

Consequences

- Advantage in ID-based crypto if recipient is not in system or sender wants to force use of a fresh key (other applications possible).
- No need for PKI, can avoid need for authentication.
- Set-up requires a trusted authority (TA) which can compute the secret key for a given public key.

ID-based cryptography II

- Let $H : \{0,1\}^* \rightarrow G_1'$ be hash function.
- Master secret key of TA is s, public key is $P_{pub} = [s]P$.
- Public key of ID represented by string ID is $H(ID) \in G_1'$.
- Secret key $S' = [s]H(ID) \in G_1'$ computable only by TA.
- Idea is that from

  $$P_{pub}, H(ID) \text{ and } k$$

  or from

  $$[k]P, \text{ and } [s]H(ID)$$

  both parties can compute

  $$e([k]P, [s]H(ID)) = (e(P, H(ID)))^{ks} = (e([s]P, H(ID)))^k.$$

Security assumptions

(Just a few from the zoo.)

- Clearly these systems require that the DLP is hard in the groups.
- Additionally we define the following computational and decisional problems. To ease notation let $G_1 = G_1'$ and $g = e(P, P)$.
  - Computational Bilinear Diffie-Hellman Problem (CBDHP):
    Compute $g^{s_A s_B s_C}$ given $[s_A]P, [s_B]P, [s_C]P$, and P
  - Decisional Bilinear Diffie-Hellman Problem (DBDHP):
    Given $P, [s_A]P, [s_B]P, [s_C]P$ and g^r decide whether $g^r = g^{s_A s_B s_C}$.

Divisor class groups

(Arithmetic on hyperelliptic curves)

Example: Hyperelliptic Curves

Affine equation of hyperelliptic curve of genus g (with $\mathbb{F}_q$-rational Weierstraß-point at infinity)

$$C : y^2 + h(x)y = f(x)$$

$h(x), f(x) \in \mathbb{F}_q[x]$, f monic, $\deg f = 2g + 1, \deg h \leq g$
non singular, i. e. not both partial derivatives

$(2y + h(x)$ and $h'(x)y - f'(x))$

vanish in any in $(a, b) \in C/\overline{\mathbb{F}_q}$

Group

Use Divisors, i. e. finite sums of points (elements of the free abelian group), of degree zero

$$\sum_{P \in C(\overline{\mathbb{F}_q})} n_P P, \; n_P \in \mathbb{Z}$$

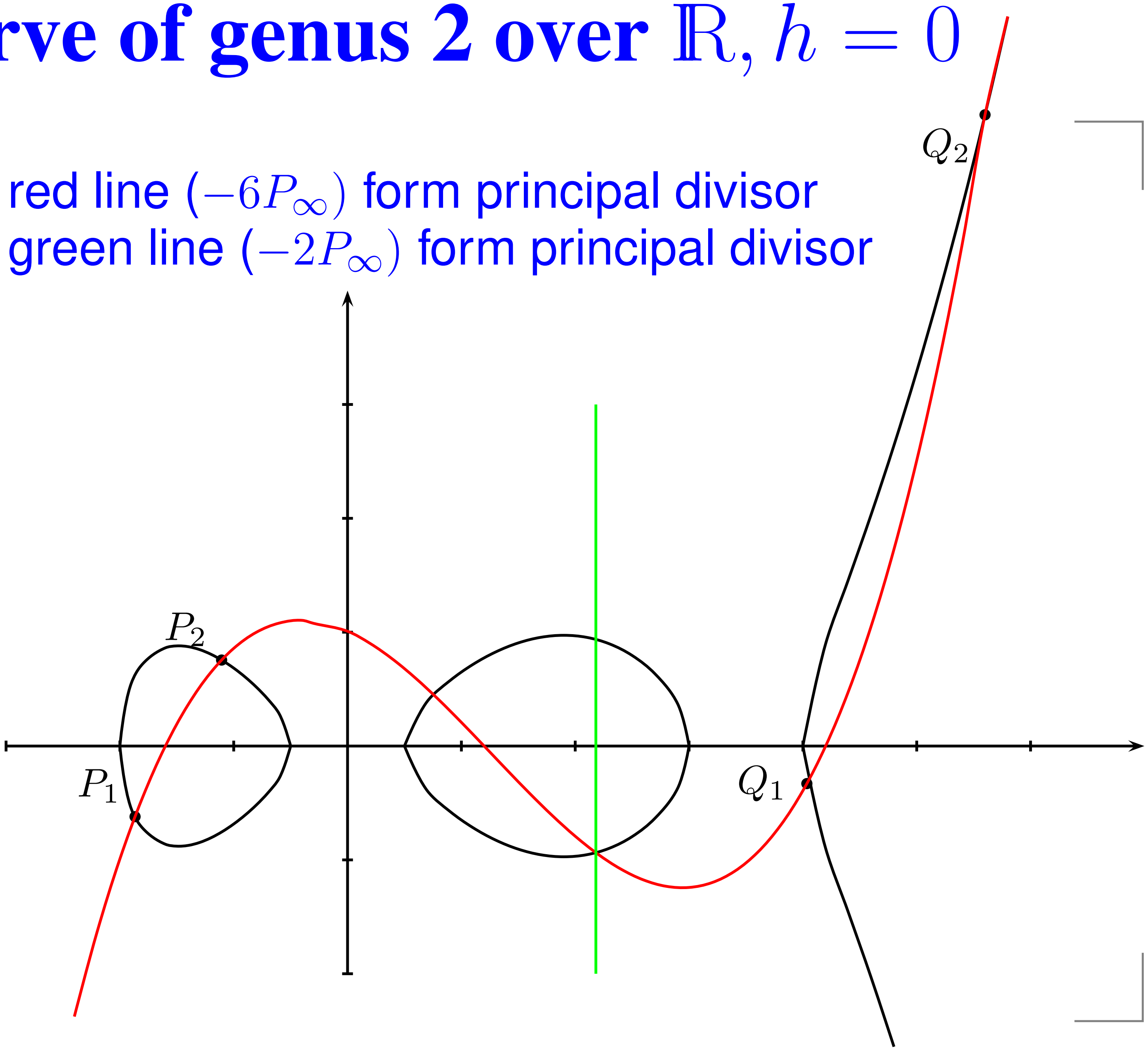
with $\sum_{P \in C(\overline{\mathbb{F}_q})} n_P = 0$ a $n_P = 0$ for almost all P.
One associates the divisor

$$\mathrm{div}(F) = \sum_{P \in C(\overline{\mathbb{F}_q})} v_P(F) P$$

to the function $F \in \mathbb{F}_q(C)$, where v_P is a valuation in the point P. Such divisors are called principal divisors Princ_C.
One can show that they have degree zero.

Curve of genus 2 over $\mathbb{R}, h = 0$

points on red line $(-6P_\infty)$ form principal divisor
points on green line $(-2P_\infty)$ form principal divisor



Divisor class group

Consider the factor group of the group of divisors of degree zero Div_C^0 modulo the principal divisors. This way one constructs the divisor class group of degree zero.

$$\mathrm{Pic}_C^0 = \mathrm{Div}_C^0/\mathrm{Princ}_C.$$

The $\mathbb{F}_q$-rational group elements $\mathrm{Pic}_C^0(\mathbb{F}_q)$ are those which remain fixed under $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

This construction provides an isomorphism between $\mathrm{Pic}_C^0(\mathbb{F}_q)$ and the $\mathbb{F}_q$-rational points of the Jacobian of C.

Representation of group elements

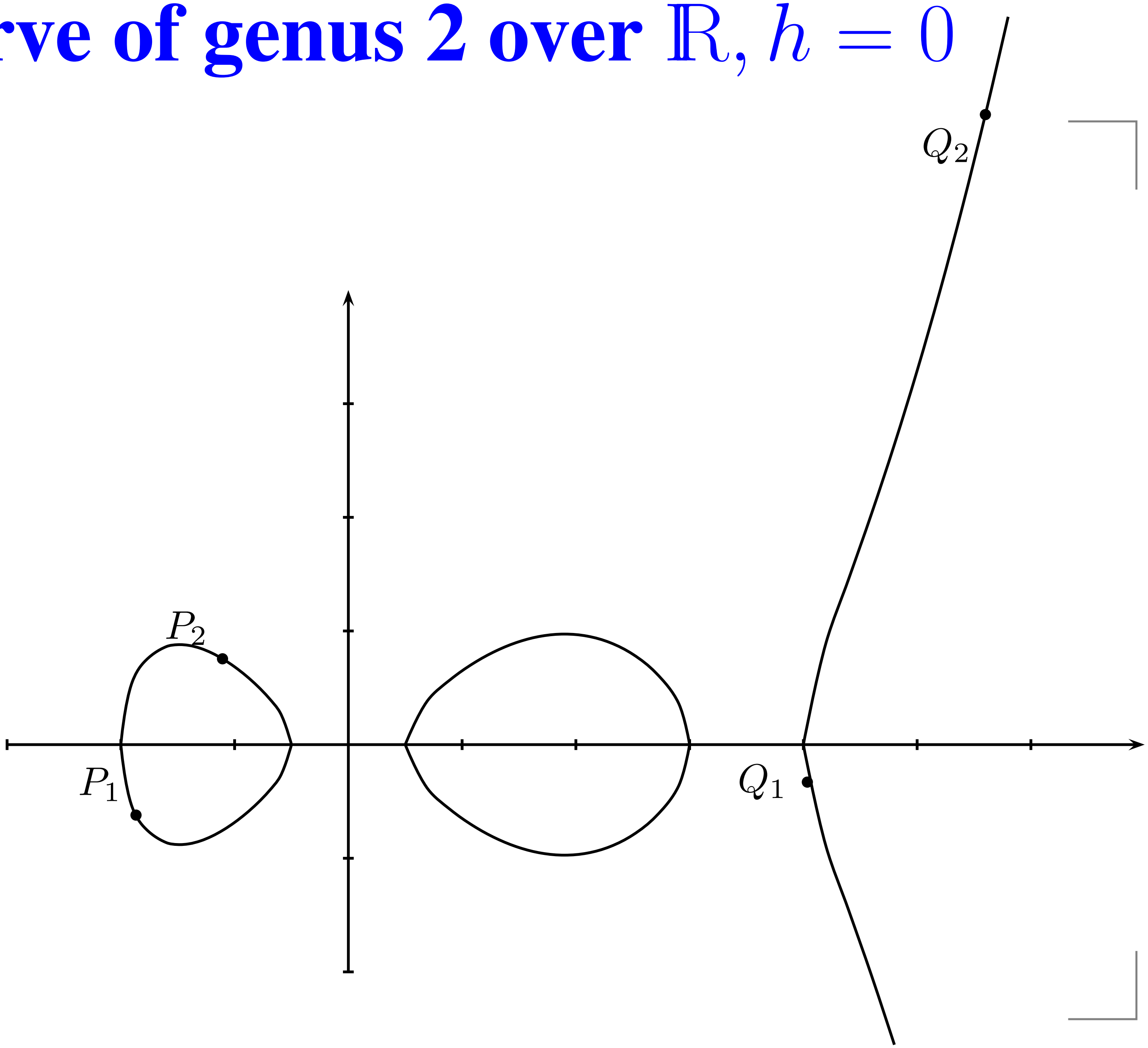
Riemann-Roch allows to find a unique reduced representation by means of a divisor of degree zero with $n \leq g$

$$\bar{D} = \sum_{\substack{i=1 \\ P_i \in C(\overline{\mathbb{F}_q}) \setminus \{P_\infty\}}}^{n} P_i - nP_\infty$$

and $P_i \neq \iota(P_j)$ for $i \neq j$, where ι is the hyperelliptic involution.

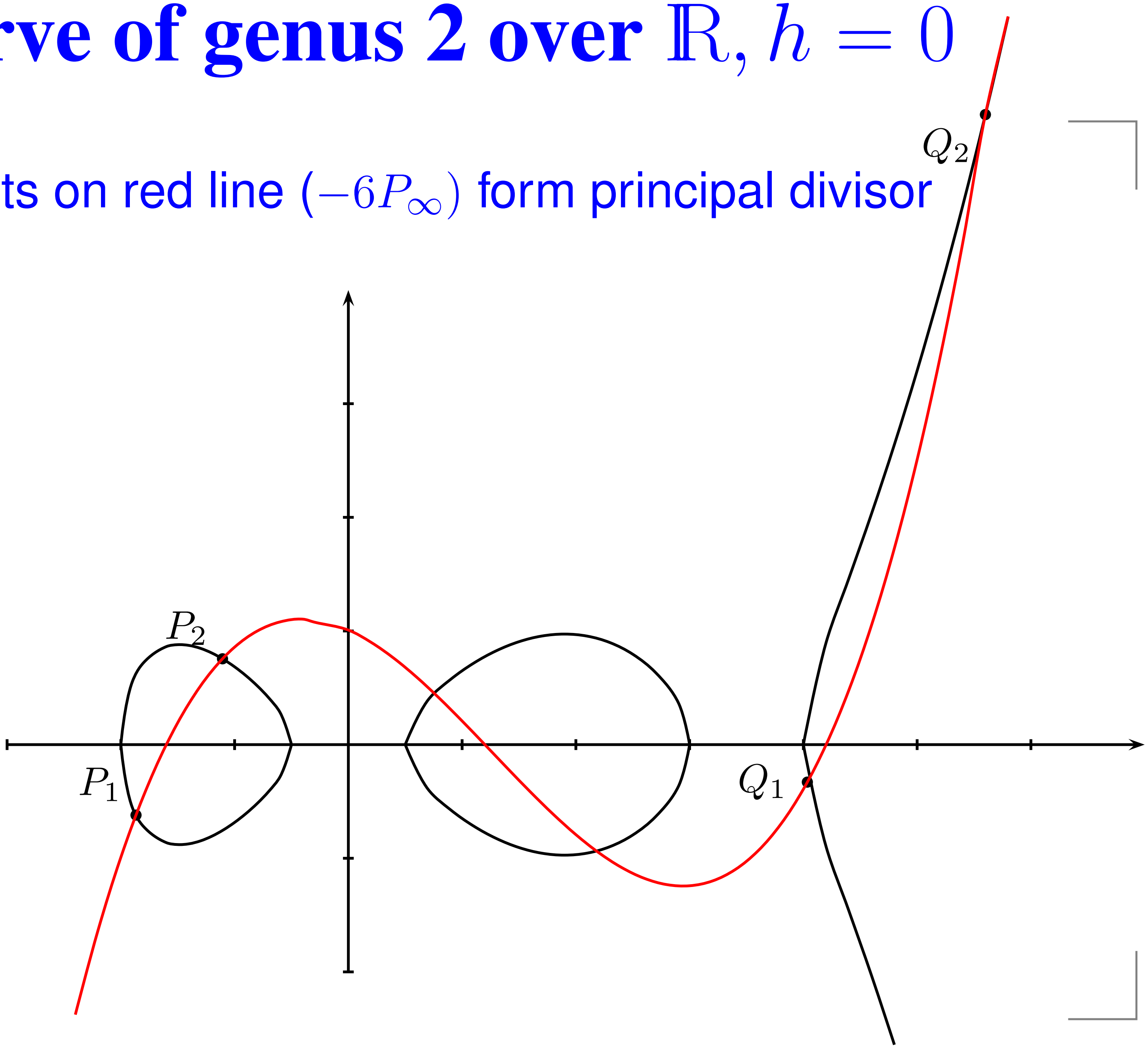
If $\bar{D}$ is defined over $\mathbb{F}_q$, the extension degree of the field of definition of the P_l is bounded; e. g. for $g = 2$ each point is defined over $\mathbb{F}_{q^2}$.

Curve of genus 2 over $\mathbb{R}, h = 0$



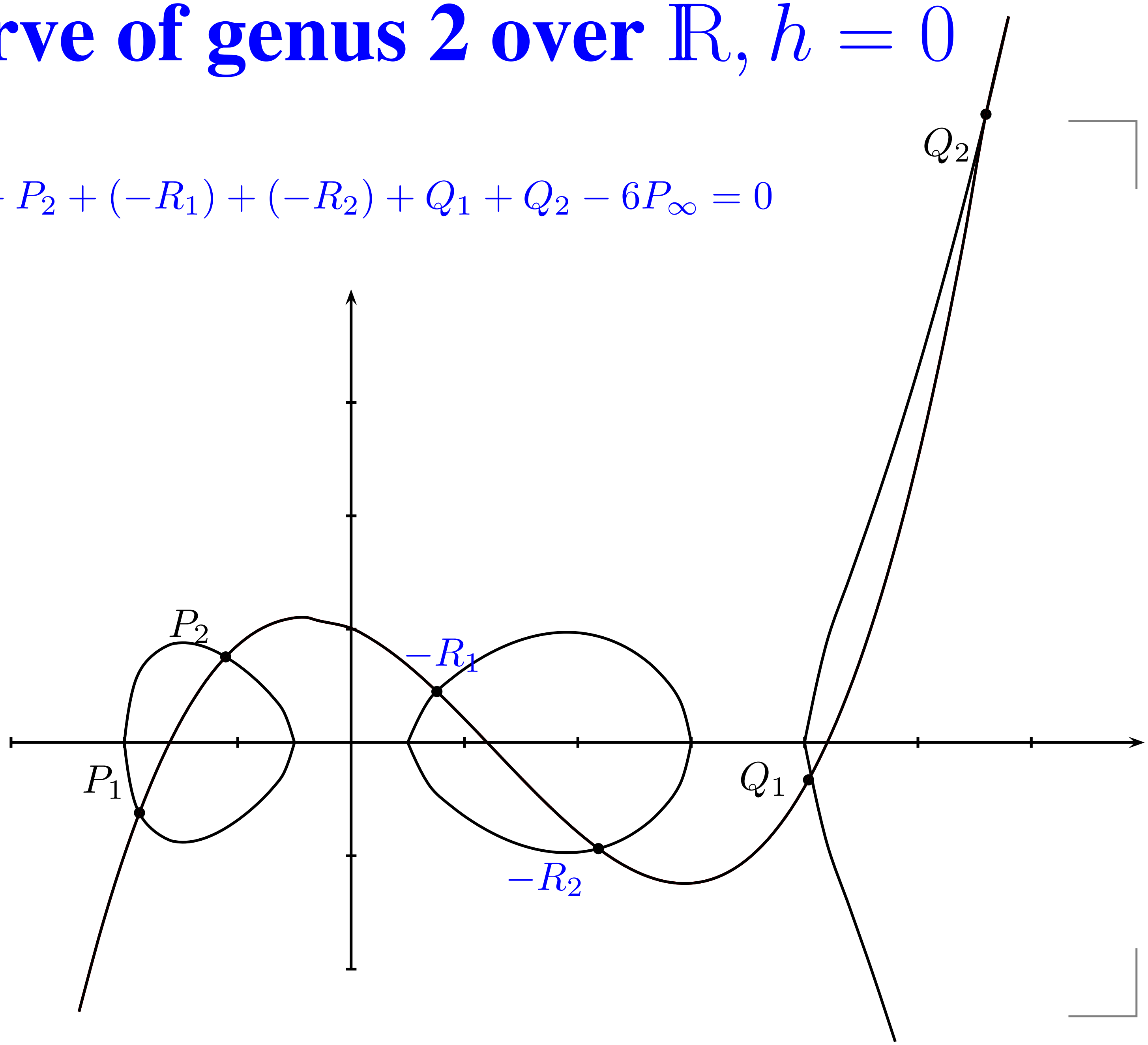
Curve of genus 2 over $\mathbb{R}$, $h = 0$

points on red line $(-6P_\infty)$ form principal divisor



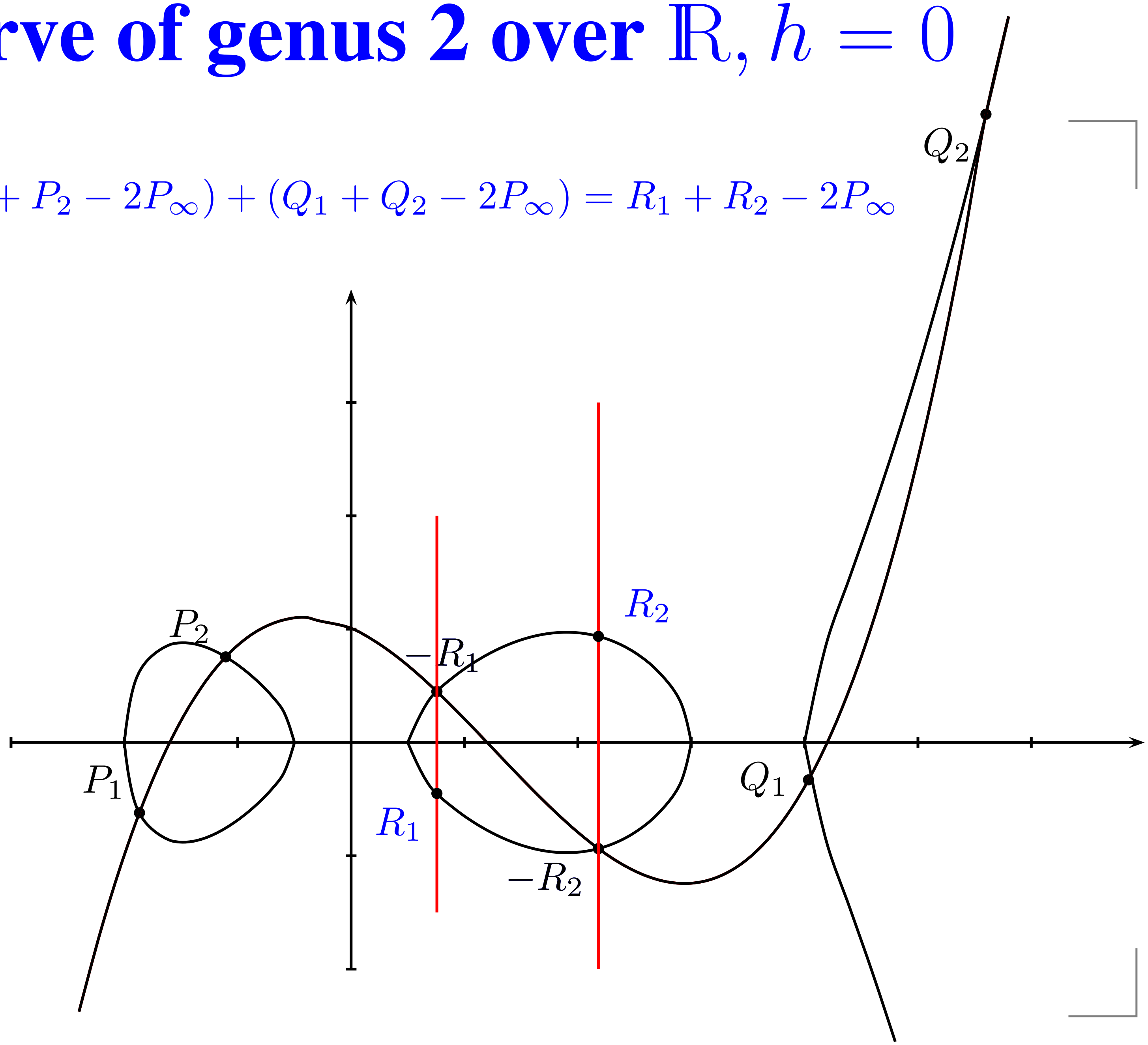
Curve of genus 2 over $\mathbb{R}, h = 0$

$$P_1 + P_2 + (-R_1) + (-R_2) + Q_1 + Q_2 - 6P_\infty = 0$$



Curve of genus 2 over $\mathbb{R}$, $h = 0$

$$(P_1 + P_2 - 2P_\infty) + (Q_1 + Q_2 - 2P_\infty) = R_1 + R_2 - 2P_\infty$$



Representation – elliptic curves

By reduction modulo principal divisors (lines) one can thus reduce any divisor to just $P - P_\infty$ or the neutral element.

The isomorphism

$$\mathrm{Pic}^0_E(\mathbb{F}_{q^k}) \longrightarrow E(\mathbb{F}_{q^k}), \qquad \begin{array}{rcl} P - P_\infty & \mapsto & P \\ 0 & \mapsto & P_\infty \end{array}$$

shows that the above construction gives a group on the affine points of E together with the point at infinity.

Pairings

Prerequisites

We want to define pairings

$$G_1 \times G_1' \to G$$

preserving the group structure.

The pairings we use map to the multiplicative group of a finite extension field $\mathbb{F}_{q^k}$. To embed the points of order ℓ the embedding degree k has to be chosen large enough so that the ℓ-th roots of unity are in $\mathbb{F}_{q^k}^*$.

So choose

$$k \text{ minimal with } \ell \mid q^k - 1.$$

Tate-Lichtenbaum pairing I

$\mathrm{Pic}^0_C(\mathbb{F}_{q^k})[\ell]$: divisor classes on C of order ℓ defined over $\mathbb{F}_{q^k}$.

$\bar{D}_1 \in \mathrm{Pic}^0_C(\mathbb{F}_{q^k})[\ell] \Rightarrow \exists F_{D_1}$ such that $\ell D_1 \sim \mathrm{div}(F_{D_1})$, where D_1 represents the class $\bar{D}_1$.

Let $\bar{D}_2 \in \mathrm{Pic}^0_C(\mathbb{F}_{q^k})$ be represented by D_2 with

$$\mathrm{support}(D_2) \cap \mathrm{support}(D_1) = \emptyset.$$

Tate-Lichtenbaum pairing

$$T_\ell(\bar{D}_1, \bar{D}_2) = F_{D_1}(D_2) = \frac{\prod_{i=1}^n F_{D_1}(P_i)}{\prod_{j=1}^n F_{D_1}(Q_j)}$$

for $D_2 = \sum_{i=1}^n P_i - \sum_{j=1}^n Q_j$.

Tate-Lichtenbaum pairing II

This

$$T_\ell(\bar{D}_1, \bar{D}_2) = F_{D_1}(D_2)$$

defines a bilinear and non-degenerate map

$$T_\ell : \mathrm{Pic}^0_C(\mathbb{F}_{q^k})[\ell] \times \mathrm{Pic}^0_C(\mathbb{F}_{q^k})/\ell\mathrm{Pic}^0_C(\mathbb{F}_{q^k}) \longrightarrow \mathbb{F}^*_{q^k}/\mathbb{F}^{*\ell}_{q^k}$$

as ℓ-folds are in the kernel of T_ℓ.
Namely, if $\bar{D}_2 = [\ell]\bar{D}_3$ then

$$F_{D_1}(D_2) = F_{D_1}(D_3)^\ell = 1.$$

To achieve unique value in $\mathbb{F}_{q^k}$ rather than class do final exponentiation

$$\tilde{T}_\ell = T_\ell(\bar{D}_1, \bar{D}_2)^{(q^k-1)/\ell}.$$

Tate-Lichtenbaum pairing III

For elliptic curves use isomorphism

$$\mathrm{Pic}^0_E(\mathbb{F}_{q^k}) \cong E(\mathbb{F}_{q^k})$$

to define pairing on points $T_\ell(P,Q)$, with

$$D_1 = P - P_\infty, D_2 = (Q \oplus R) - R$$

for some R.

Build F iteratively by Miller's algorithm (double-and-add).

Often

$$T_\ell : E(\mathbb{F}_{q})[\ell] \times E(\mathbb{F}_{q^k})/\ell E(\mathbb{F}_{q^k}) \rightarrow \mathbb{F}^*_{q^k}/\mathbb{F}^{*\ell}_{q^k}.$$

Miller's algorithm

In: $\ell = \sum_{i=0}^{n-1} \ell_i 2^i$, $P, Q \oplus R, R$
Out: $T_\ell(P, Q)$

1. $T \leftarrow P$, $F \leftarrow 1$
2. for $i = n-2$ downto 0 do
   (a) Calculate lines l and v in doubling $[2]T$
   $T \leftarrow [2]T$
   $F \leftarrow F^2 \cdot l(Q \oplus R)v(R)/(l(R)v(Q \oplus R))$
   (b) if $\ell_i = 1$ then
   Calculate lines l and v in addition $T \oplus P$
   $T \leftarrow T \oplus P$
   $F \leftarrow F \cdot l(Q \oplus R)v(R)/(l(R)v(Q \oplus R))$
3. return F

Weil pairing

For an elliptic curve E define

$$\begin{aligned} W_\ell : E(\overline{\mathbb{F}}_q)[\ell] \times E(\overline{\mathbb{F}}_q)[\ell] &\rightarrow \mu_\ell \\ (P, Q) &\mapsto \frac{F_{P-P_\infty}(D_Q)}{F_{Q-P_\infty}(D_P)} \end{aligned}$$

where μ_ℓ is the multiplicative groups of the ℓ-th roots of unity in the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$.
Obviously, $W_\ell(P, P) = 1$.

Weil pairings can be seen as two-fold application of the Tate-Lichtenbaum pairing, note $Q \in E(\mathbb{F}_{q^k})$.

Needs full group of order ℓ in $E(\mathbb{F}_{q^k})$, if $k = 1$ then the Weil pairing is trivial & one needs to use larger field.

Efficient computation

k minimal

Observation:
k is minimal with $\ell \mid q^k - 1$, so final exponentiation $T_\ell(\bar{D}_1, \bar{D}_2)^{(q^k-1)/\ell}$ removes all terms from subfields.

- Choose $R \in E(\mathbb{F}_q)$, computation of $l(R)$ and $v(R)$ not needed.
- Choose $R = P_\infty$,

$$\tilde{T}_\ell(P, Q) = F_{P-P_\infty}(Q)^{(q^k-1)/\ell}.$$

- If protocol allows, choose $Q = (x_Q, y_Q) \in \mathbb{F}_{q^k}$ with x in subfield, this avoids computation of $v(Q)$. This means no extra division!

Improvements to Miller

Observation:
Let $N = c\ell$ for some c, then

$$\tilde{T}_\ell(P,Q) = T_\ell(P,Q)^{(q^k-1)/\ell} = T_N(P,Q)^{(q^k-1)/N}.$$

- Other addition chains can be used in Miller's algorithm, e.g. NAF.
  Other bases might be good depending on characteristic of the field.
- Use multiple N with low Hamming weight (in p-adic representation).
  For supersingular curves choosing N as number of points gives fast final exponentiation.

Special curves

Duursma-Lee and Barreto-Galbraith-O hEigeartaigh-Scott obtain further speed-up by choosing supersingular curves of genus $(p-1)/2$ over $\mathbb{F}_{p^r}$.

Main ideas are to use p as basis in addition chain $\Rightarrow$ much shorter chain and take into account that each order divides p^r+1.

Key words are “eta-pairing” and “eta-T-pairing”.

Recent paper ePrint 2006/110 by Hess, Smart, and Vercauteren introduces ate-pairing.

Special divisors

joint work with G. Frey.

- Useful if no map $J_C(\mathbb{F}_q) \mapsto J_C(\mathbb{F}_{q^k})$ given.
- Katagi, Kitamura, Akishita, and Takagi suggested to use special divisors $\bar{D} = P - P_\infty$ as basis for DL system, also for $g > 1$
  $\Rightarrow$ additions involving special divisor much cheaper.
- Use subset of special divisors in $J_C(\mathbb{F}_{q^k})$ as second argument of the pairing – or even restricted to points with x-coordinate in subfield.
- Evaluation of functions only in one point much faster.
- Speed-up grows with genus!
- For non-degeneracy etc. see ANTS 2006 proceedings.

Supersingular and ordinary

Definition

Let E be an elliptic curve defined over $\mathbb{F}_q, q = p^r$.
E is supersingular if

- $E[p^s](\overline{\mathbb{F}}_q) = \{P_\infty\}$.
- $t \equiv 0 \bmod p$.
- End_E is order in quaternion algebra.

Otherwise it is ordinary and one has $E[p^s](\overline{\mathbb{F}}_q) = \mathbb{Z}/p^s\mathbb{Z}$.
These statements hold for all s if they hold for one.

Example:

$$y^2 + y = x^3 + a_4 x + a_6 \text{ over } \mathbb{F}_{2^r}$$

is supersingular, as a point of order 2 would satisfy

$$y_P = y_P + 1$$

which is impossible.

Distortion map I

For supersingular curves it is possible to find maps $\phi : E(\mathbb{F}_q) \to E(\mathbb{F}_{q^k})$ that map to a linearly independent subgroup, i.e.

$$T'_\ell(P,P) \neq 1 \text{ for } T'_\ell(P,P) = T_\ell(P,\phi(P)).$$

(This needs that there are independent endomorphisms, so no chance for ordinary curves).
Examples:

- $y^2 = x^3 + a_4 x$, for $p \equiv 3 \pmod 4$.
  Distortion map $(x,y) \mapsto (-x, iy)$ with $i^2 = -1$
- $y^2 = x^3 + a_6$, for $p \equiv 2 \pmod 3$.
  Distortion map $(x,y) \mapsto (jx, y)$ with $j^3 = 1, j \neq 1$,

In both cases, $\#E(\mathbb{F}_p) = p + 1$, $k = 2$.

Distortion maps II

- Over $\mathbb{F}_{2^d}$ consider
$$y^2 + y = x^3 + x + a_6, \text{ with } a_6 = 0 \text{ or } 1$$
and distortion map
$$(x, y) \mapsto (x+s^2, y+sx+t),\ s, t \in \mathbb{F}_{2^{4d}},\ s^4+s = 0,\ t^2+t+s^6+s^2 = 0.$$
$$\#E(\mathbb{F}_{2^d}) = 2^d + 1 \pm 2^{(d+1)/2},\ k = 4.$$
- Over $\mathbb{F}_{3^d}$ consider
$$y^2 = x^3 + x + a_6, \text{ with } a_6 = \pm 1$$
and distortion map
$$(x, y) \mapsto (-x + s, iy) \text{ with } s^3 + 2s + 2a_6 = 0 \text{ and } i^2 = -1.$$
$$\#E(\mathbb{F}_{3^d}) = 3^d + 1 \pm 3^{(d+1)/2},\ k = 6.$$

Application of distortion maps

- Distortion maps allow to state the protocols as using bilinear map
$$G_1 \times G_1 \to G.$$
- Scalar multiplications of input point can be performed over $\mathbb{F}_q$ and then mapped into $E(\mathbb{F}_{q^k})$, this saves computation time
$\Rightarrow$ obvious computational savings in the above protocols.
- Can be used to map on points with x-coordinate in subfield.
- Systems are easier to state.
- Really short signatures (see next slide).

Short signatures

Boneh, Lynn, and Shacham, Asiacrypt 2001

Requires hash function $H : \{0,1\}^* \to G_1$.

User secret key s, public key $[s]P$.

Signature of message m:
Compute and send $S = [s]H(m)$.

Verification:
Check if

$$e([s]P, H(m)) = e(P, S).$$

Point compression to the x-coordinate achieves signatures given by one field element, i.e. of half the length of ECDSA.

Embedding degree

Embedding degrees I

Let E be supersingular and $p \geq 5$, i.e $p > 2\sqrt{p}$.

Hasse's Theorem states $|t| \leq 2\sqrt{q}$ and E supersingular implies $t \equiv 0 \bmod p$, so $t = 0$ and

$$|E(\mathbb{F}_p)| = p + 1.$$

Obviously

$$(p+1) \mid p^2 - 1 = (p+1)(p-1)$$

so $k \leq 2$ for supersingular curves over prime fields.

Embedding degrees II

Let E be supersingular.
Menezes, Okamoto, and Vanstone show:

- in characteristic 2 we have $k \leq 4$,
- in characteristic 3 we have $k \leq 6$,
- over prime fields $\mathbb{F}_p$ with $p \geq 5$ we have $k \leq 2$,

and these bounds are attained.

Similar bounds for larger genus – no or not much growth of resulting field size versus number of points in the Jacobian.

Parameter sizes

For DLP in groups one has square-root attacks, for finite fields index calculus methods are more efficient. A common requirement is

- $\#E(\mathbb{F}_q) \sim q$: 160 bits
- q^k: 1024 bits
- optimal embedding degree would be

$$k = 1024/160 = 6.4$$

However, 6 is only attained for small characteristic and there Coppersmith methods require larger fields, cf. Page, Smart, and Vercauteren.
Difference between sizes grows $\Rightarrow$ larger k becomes optimal.

MNT curves

Find ordinary curves with small embedding degree! Thus force

$$q + 1 - t \equiv 0 \pmod{\ell} \text{ and } q^k - 1 \equiv 0 \pmod{\ell}.$$

- Original proposal by Miyaji, Nakabayashi, and Takano.
- Dupont, Enge, Morain: small k by CM
- Barreto, Lynn, Scott; Brezing, Weng;
  Curves for $k = 3, 4$ and 6.
- Generalization to co-factors, $k = 5, 10, 12$
  Galbraith, McKee, Valença.
- Family of curves with $k = 12$ by Barreto and Naehrig.
- Family of curves with $k = 10$ by D. Freeman.

Supersingular vs. ordinary

Supersingular:

- pro: known, speed ups by choosing $N = c\ell$, eta-T-pairing, distortion maps
- con: only small k, far too large field size for prime fields, small number of curves

Ordinary:

- pro: can be used over prime fields, more and larger embedding degrees possible, more (but also finite) choice
- con: no distortion maps, no reason why good multiple N should exist, no loop-shortening

Outlook

Pairings are a very active area of research, e.g. on the following tasks

- find more protocols / solve posed problems,
- speed up the computation,
- find more MNT curves,
- consider other curves, e.g. hyperelliptic curves.

Great collection at

P. S. L. M. Barreto, “The Pairing-Based Crypto Lounge”

`http://planeta.terra.com.br/informatica/paulobarreto/pblounge.html`

- Break system???

The end

Literature

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- Chapter in "Advances in Elliptic Curve Cryptography", by I. F. Blake, G. Seroussi, and N. P. Smart (Editors)
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