Cryptographic Applications of Trace Zero Varieties

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Overview

- Arithmetic on hyperelliptic curves.
- Koblitz curves
- Trace Zero varieties
  - Security
  - Efficiency
  - Applications and Timings

Arithmetic on hyperelliptic curves

Hyperelliptic Curves

Affine equation of hyperelliptic curve of genus g (with $\mathbb{F}_q$-rational Weierstraß-point at infinity)

$$C : y^2 + h(x)y = f(x)$$

$h(x), f(x) \in \mathbb{F}_q[x]$, f monic, $\deg f = 2g + 1, \deg h \leq g$
non singular, i. e. not both partial derivatives

$(2y + h(x)$ and $h'(x)y - f'(x))$

vanish in any in $(a, b) \in C/\overline{\mathbb{F}_q}$

Examples

Concerning the arithmetic properties one can consider elliptic curves as hyperelliptic curves, i. e.

$$y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6$$

is considered as curve of genus one.

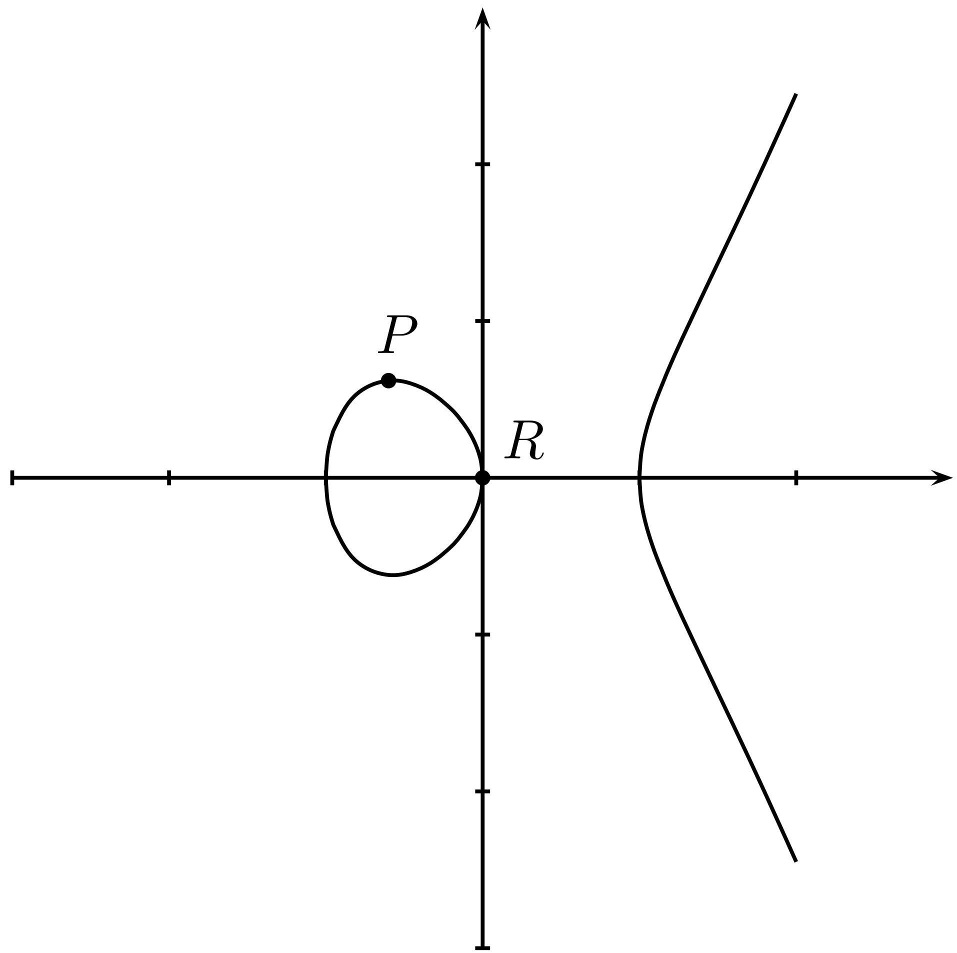
Curve of genus 2 over field of odd characteristic

$$y^2 = x^5 + f_3x^3 + f_2x^2 + f_1x + f_0,$$

provided $f(x)$ has no multiple roots.

Group Law in $E(\mathbb{R}), h = 0$

$y^2 = x^3 - x$



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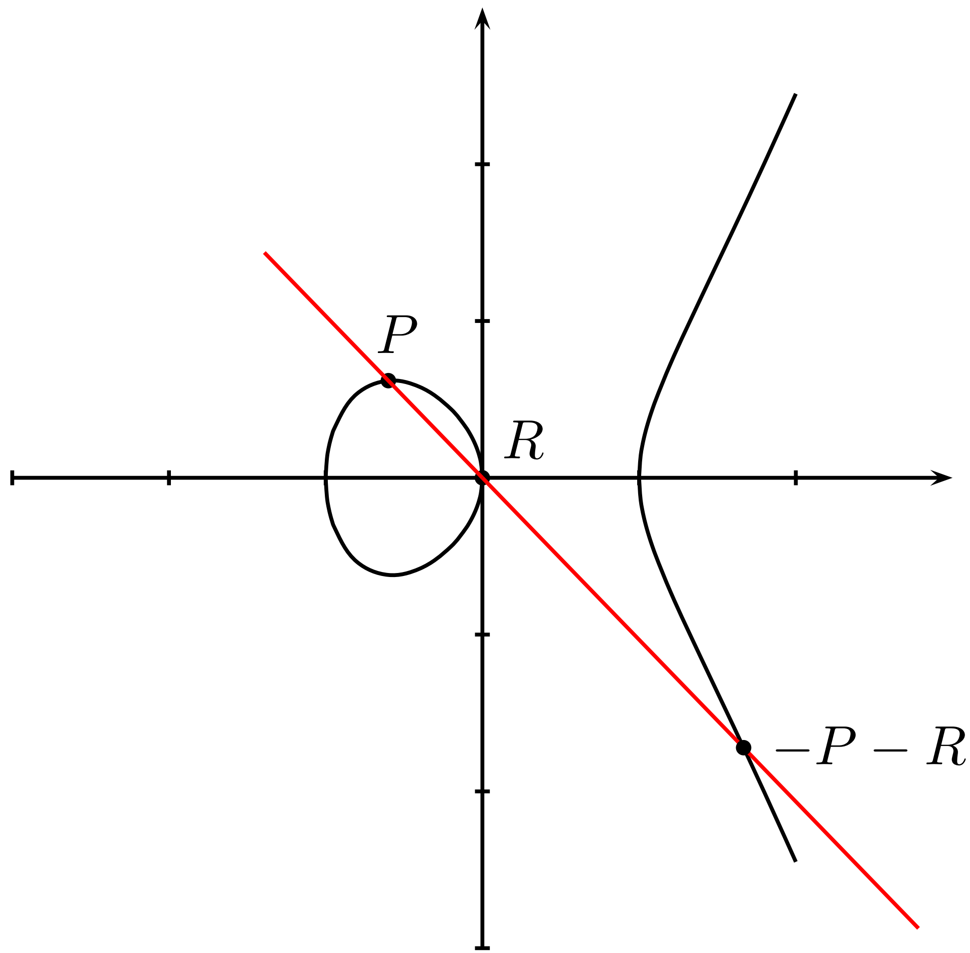

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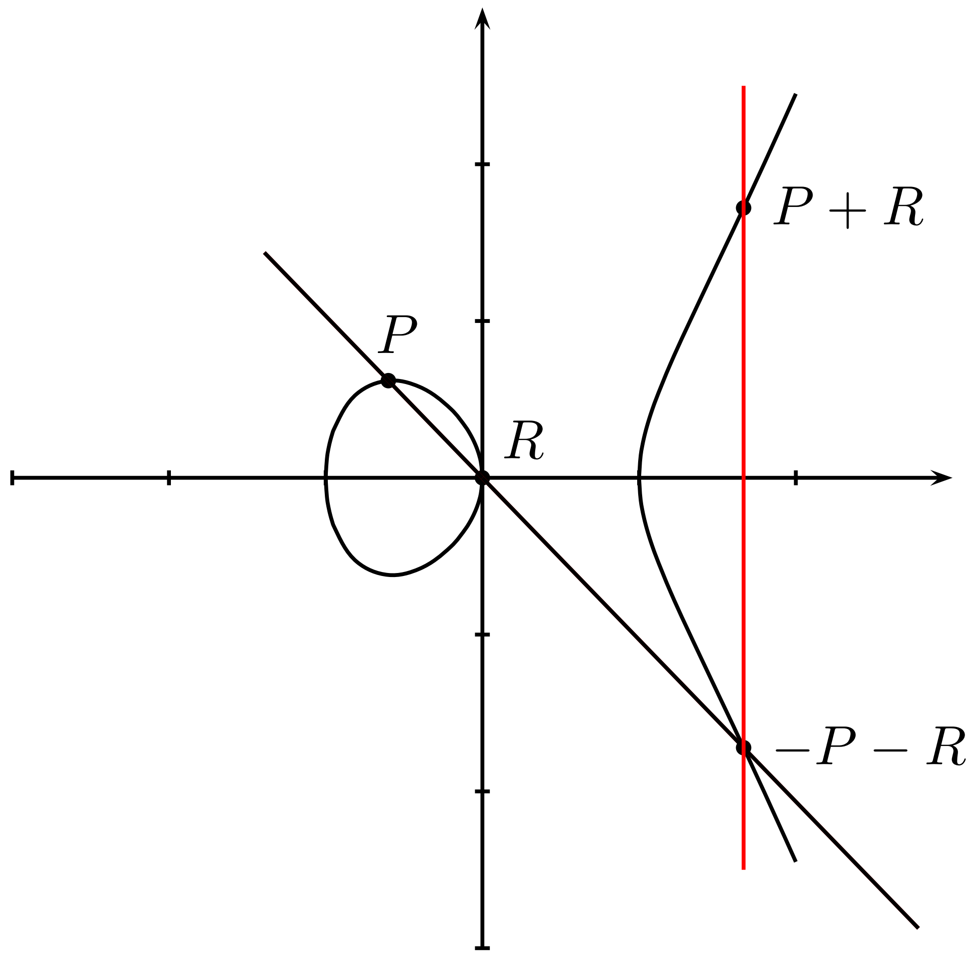
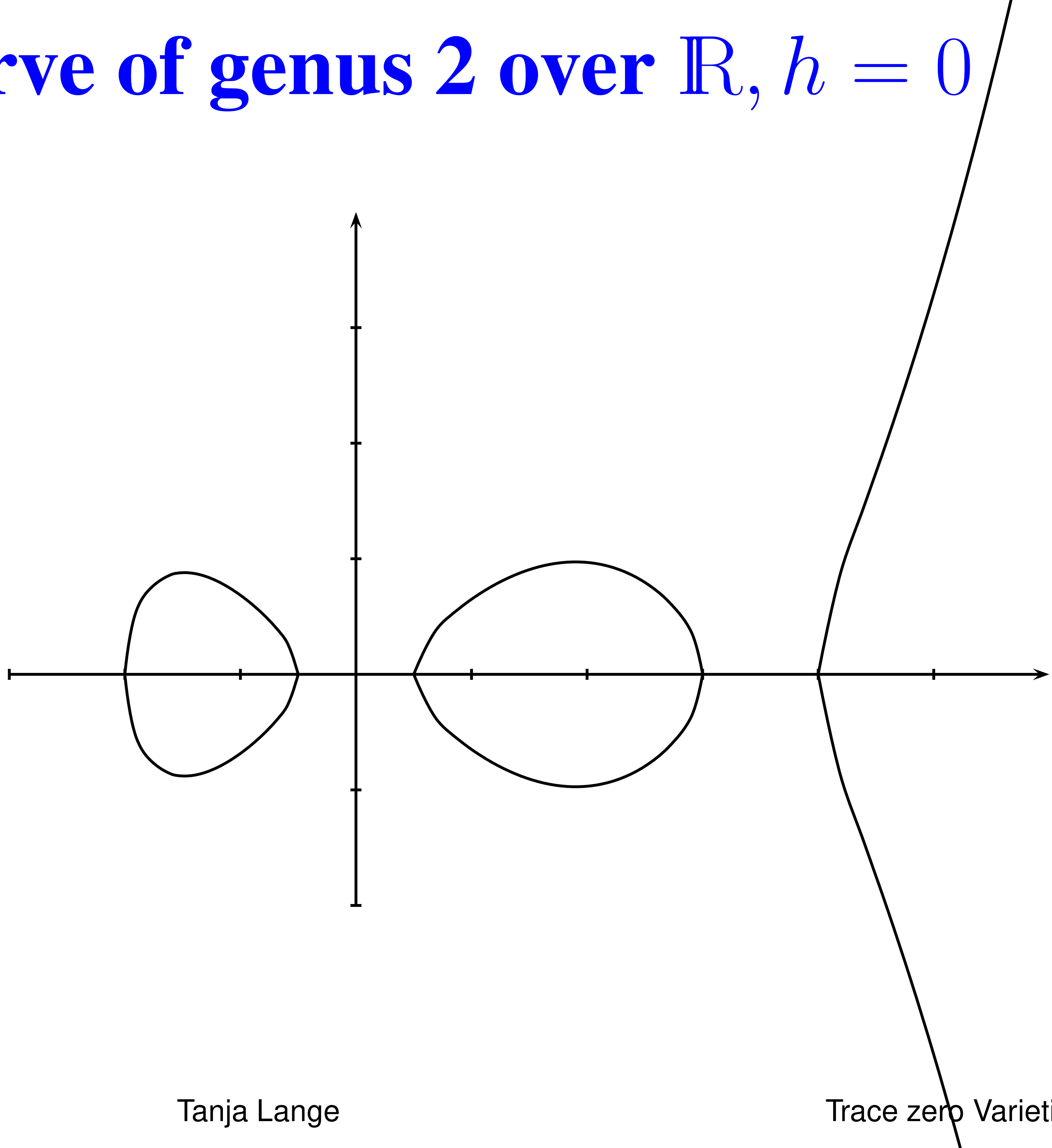
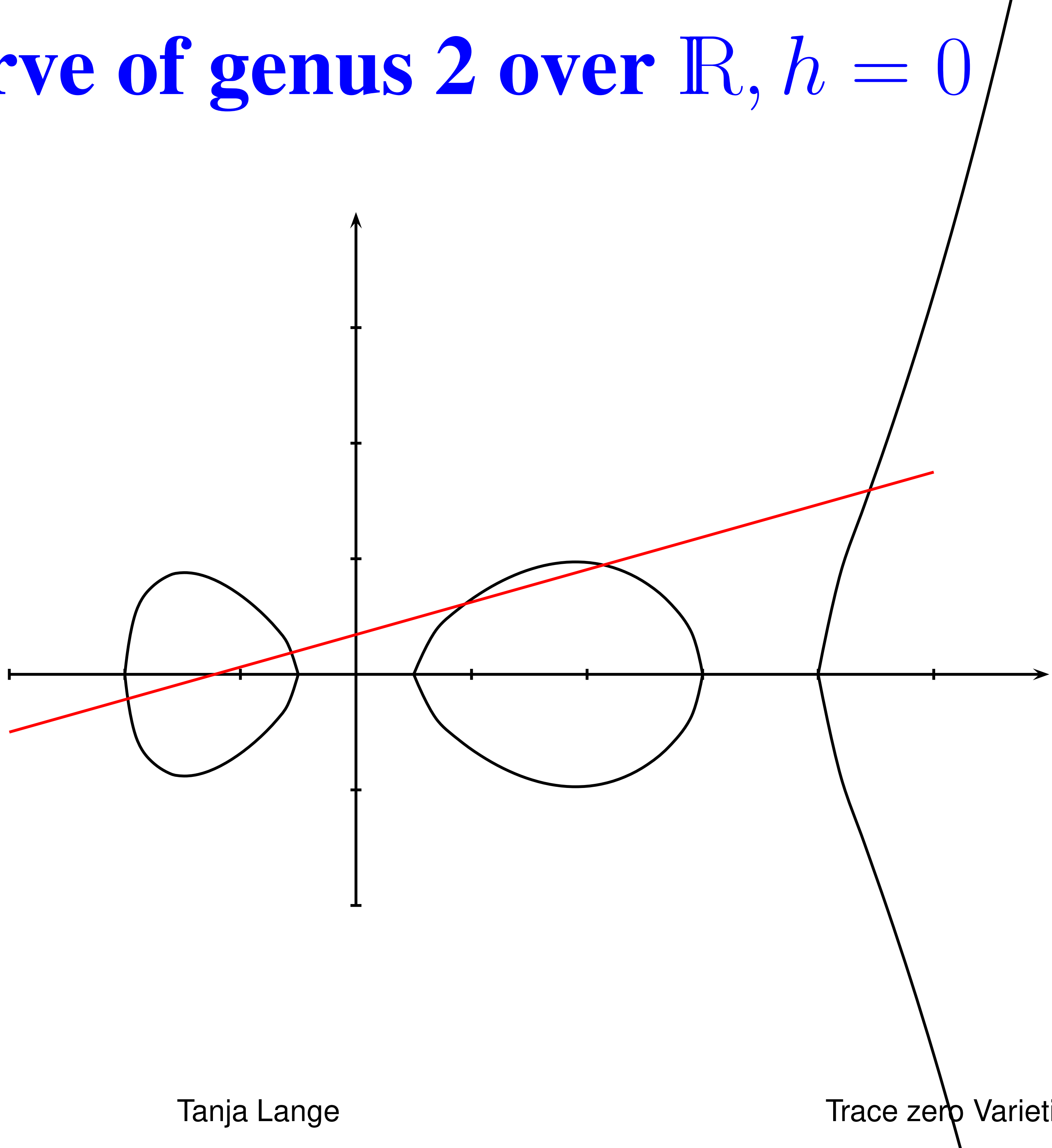


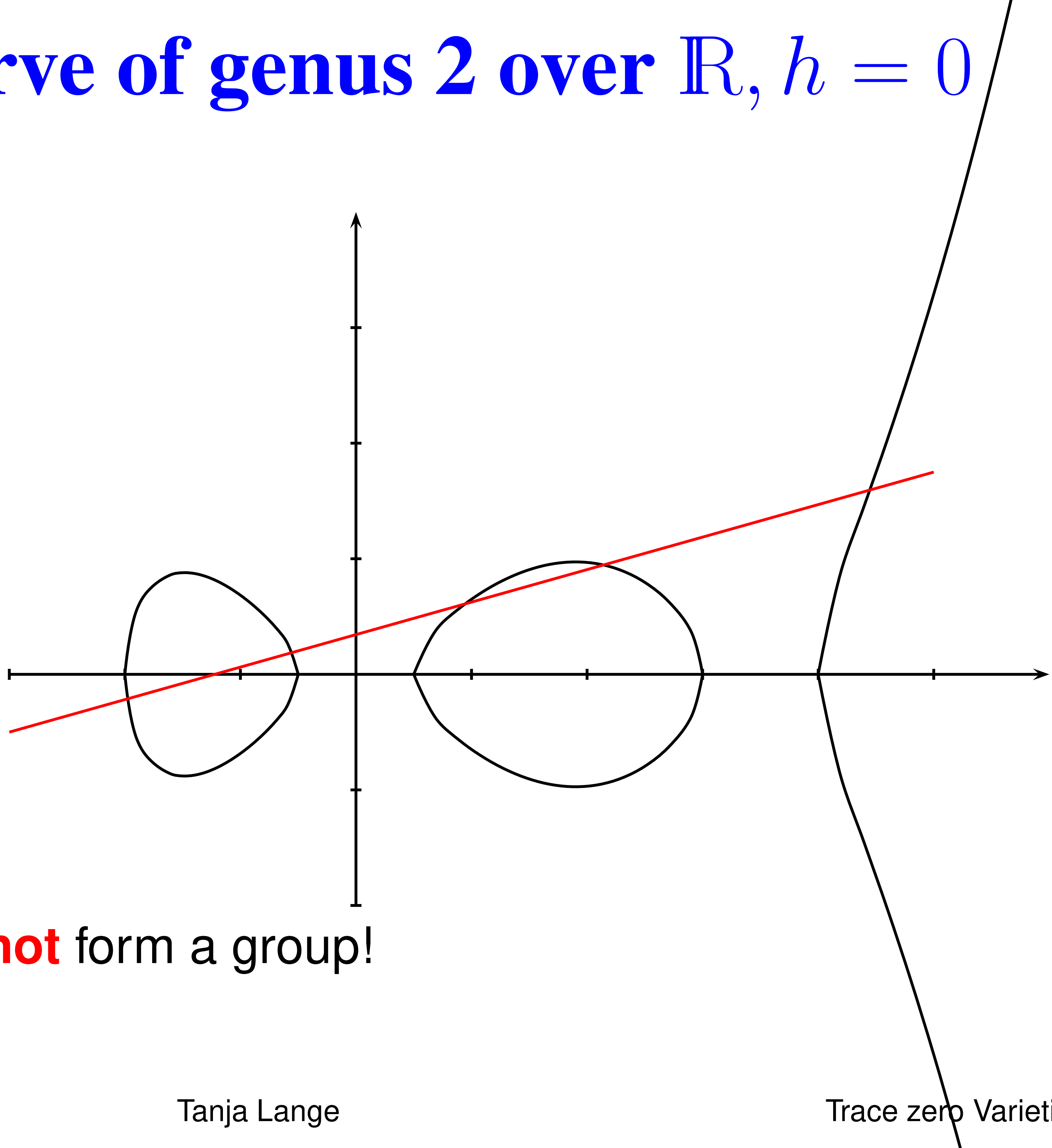
Curve of genus 2 over $\mathbb{R}, h = 0$



Curve of genus 2 over $\mathbb{R}$, $h = 0$



Curve of genus 2 over $\mathbb{R}, h = 0$



points do **not** form a group!

Group

Use **Divisors**, i. e. finite sums of points (elements of the free abelian group), of degree zero

$$\sum_{P \in C(\overline{\mathbb{F}_q})} n_P P,\ n_P \in \mathbb{Z}$$

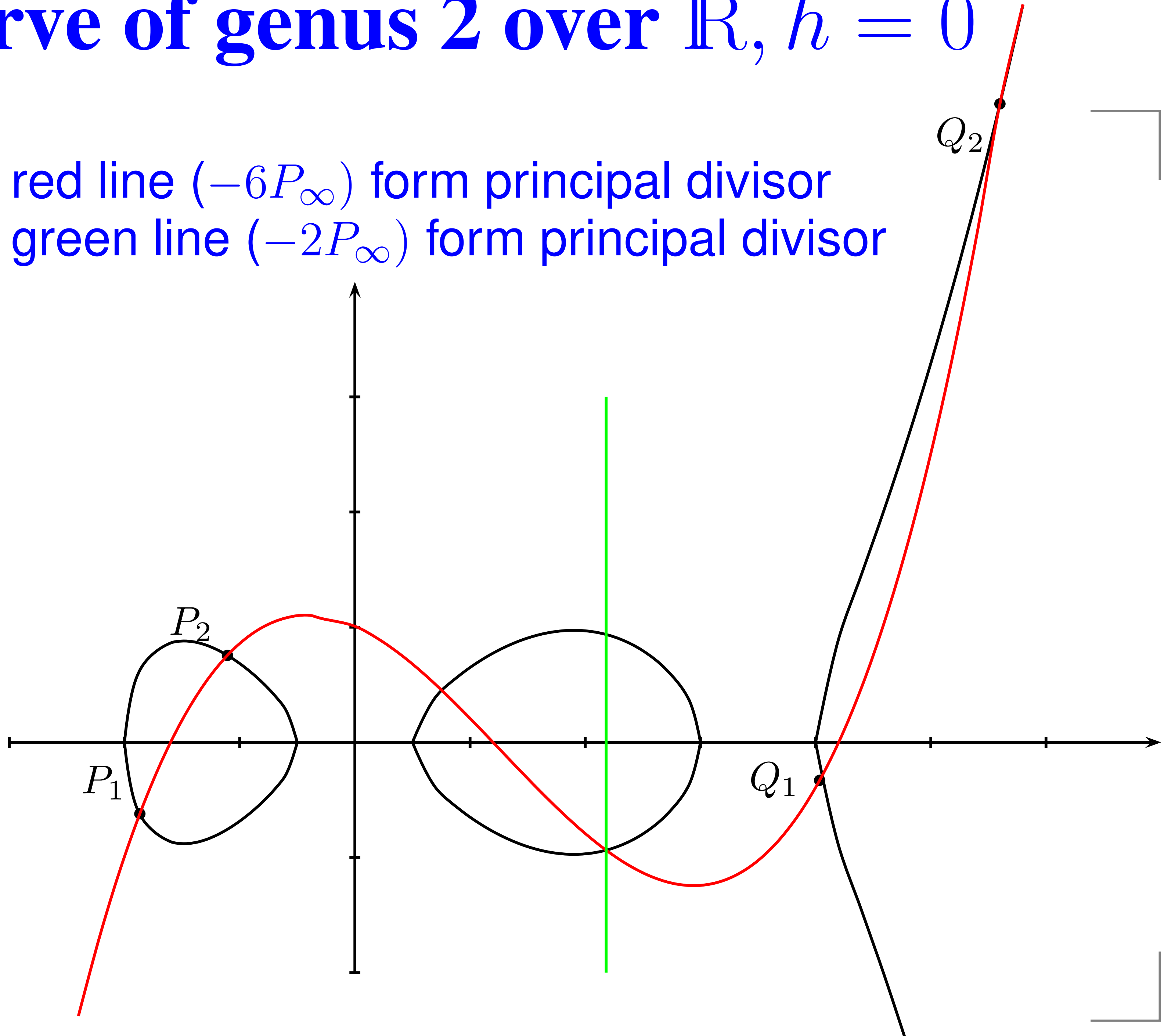
with $\sum_{P \in C(\overline{\mathbb{F}_q})} n_P = 0$ a $n_P = 0$ for almost all P.
One associates the divisor

$$\operatorname{div}(F) = \sum_{P \in C(\overline{\mathbb{F}_q})} v_P(F) P$$

to the function $F \in \mathbb{F}_q(C)$, where v_P is a valuation in the point P. Such divisors are called *principal divisors* Princ_C.
One can show that they have degree zero.

Curve of genus 2 over $\mathbb{R}, h = 0$

points on red line $(-6P_\infty)$ form principal divisor
points on green line $(-2P_\infty)$ form principal divisor



Divisor class group

Consider the factor group of the group of divisors of degree zero Div_C^0 modulo the principal divisors. This way one constructs the divisor class group of degree zero.

$$\mathrm{Pic}_C^0 = \mathrm{Div}_C^0/\mathrm{Princ}_C.$$

The $\mathbb{F}_q$-rational group elements $\mathrm{Pic}_C^0(\mathbb{F}_q)$ are those which remain fixed under $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

This construction provides an isomorphism between $\mathrm{Pic}_C^0(\mathbb{F}_q)$ and the $\mathbb{F}_q$-rational points of the Jacobian of C.

Representation of group elements

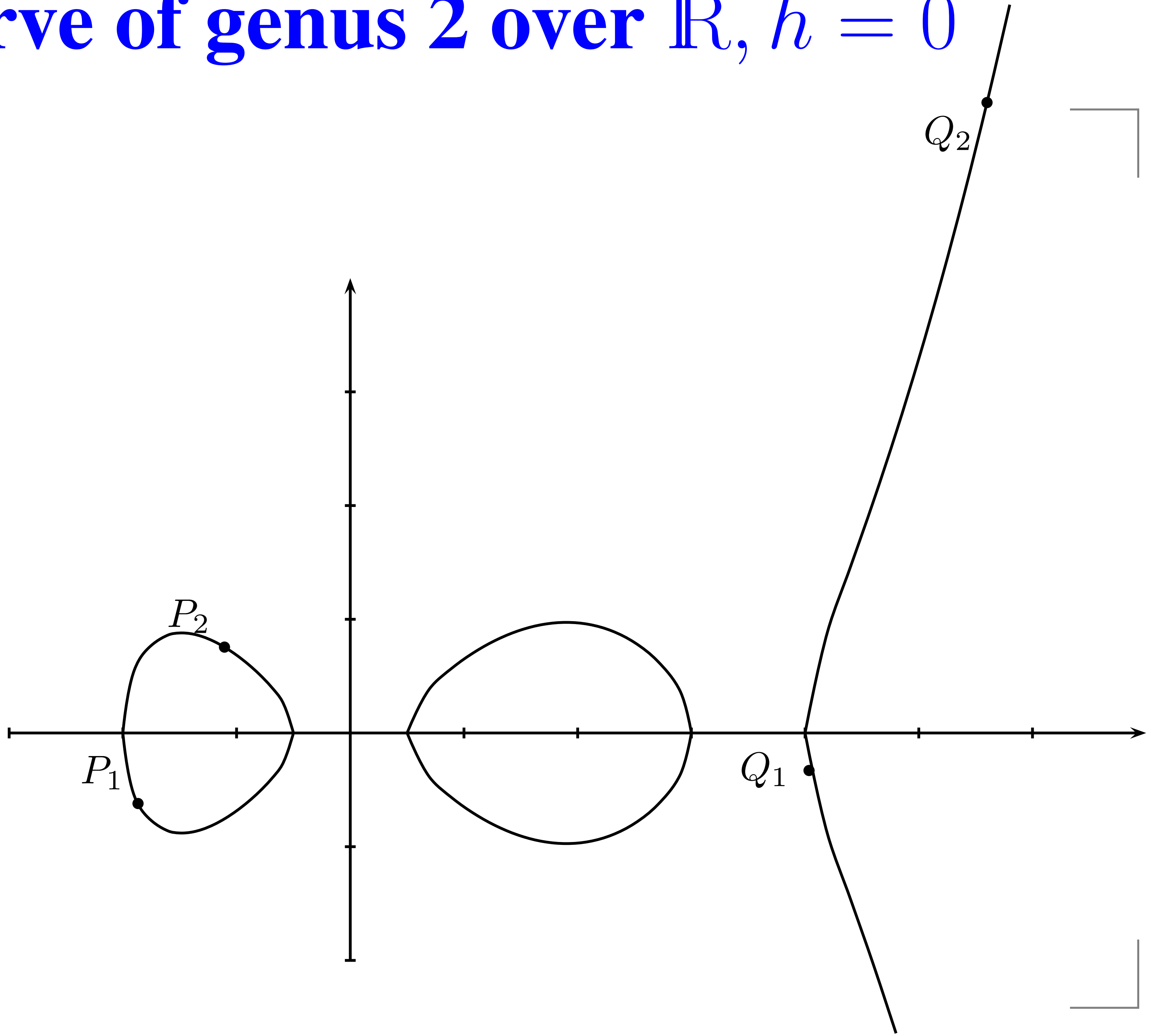
Riemann-Roch allows to find a unique reduced representation by means of a divisor of degree zero with $m \leq g$

$$\bar{D} = \sum_{\substack{i=1 \\ P_i \in C(\overline{\mathbb{F}_q}) \setminus \{P_\infty\}}}^{m} P_i - mP_\infty$$

and $P_i \neq -P_j$ for $i \neq j$.

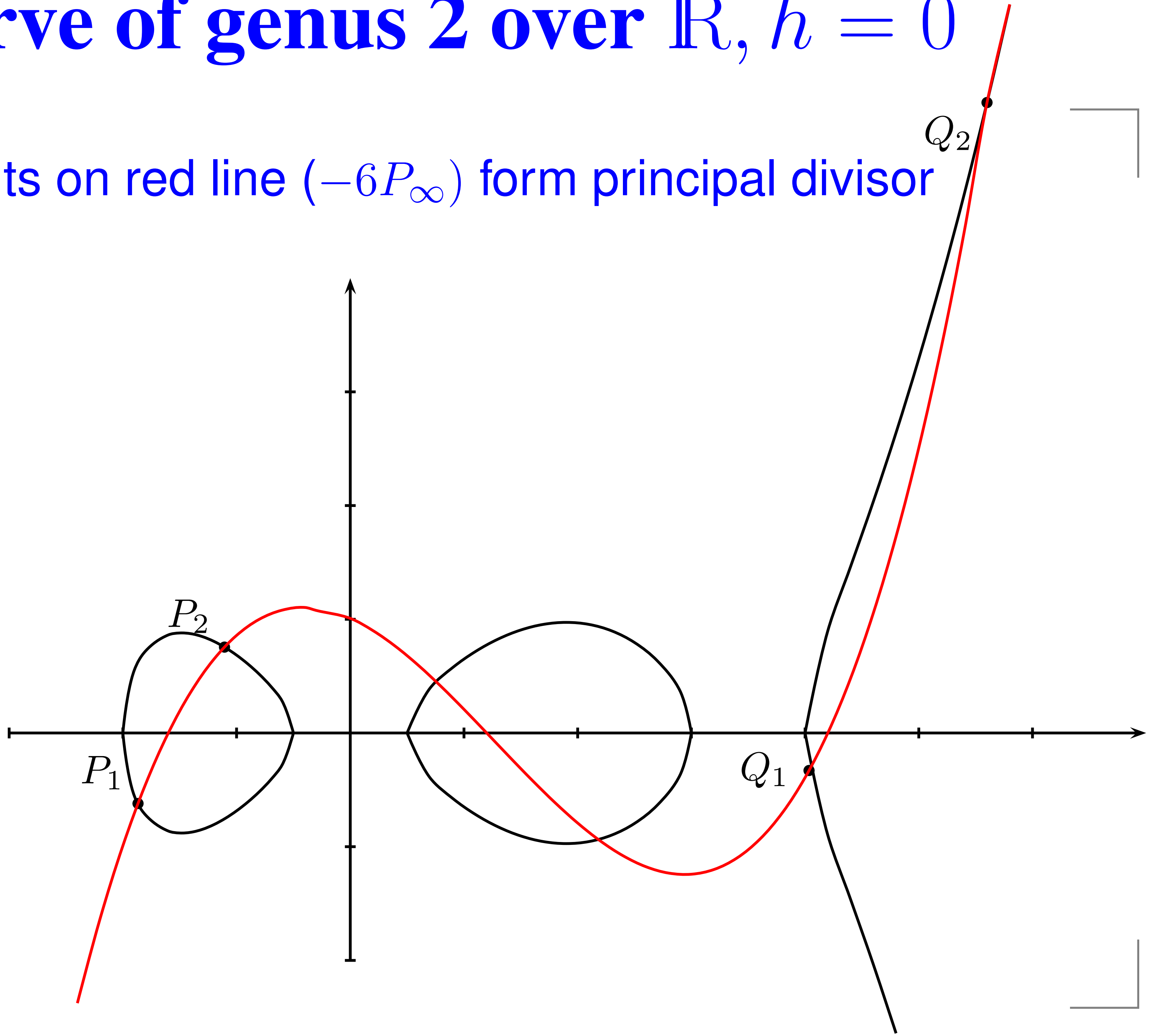
If $\bar{D}$ is defined over $\mathbb{F}_q$, the extension degree of the field of definition of the P_i is bounded, e. g. at most 2 for $g = 2$.

Curve of genus 2 over $\mathbb{R}$, $h = 0$



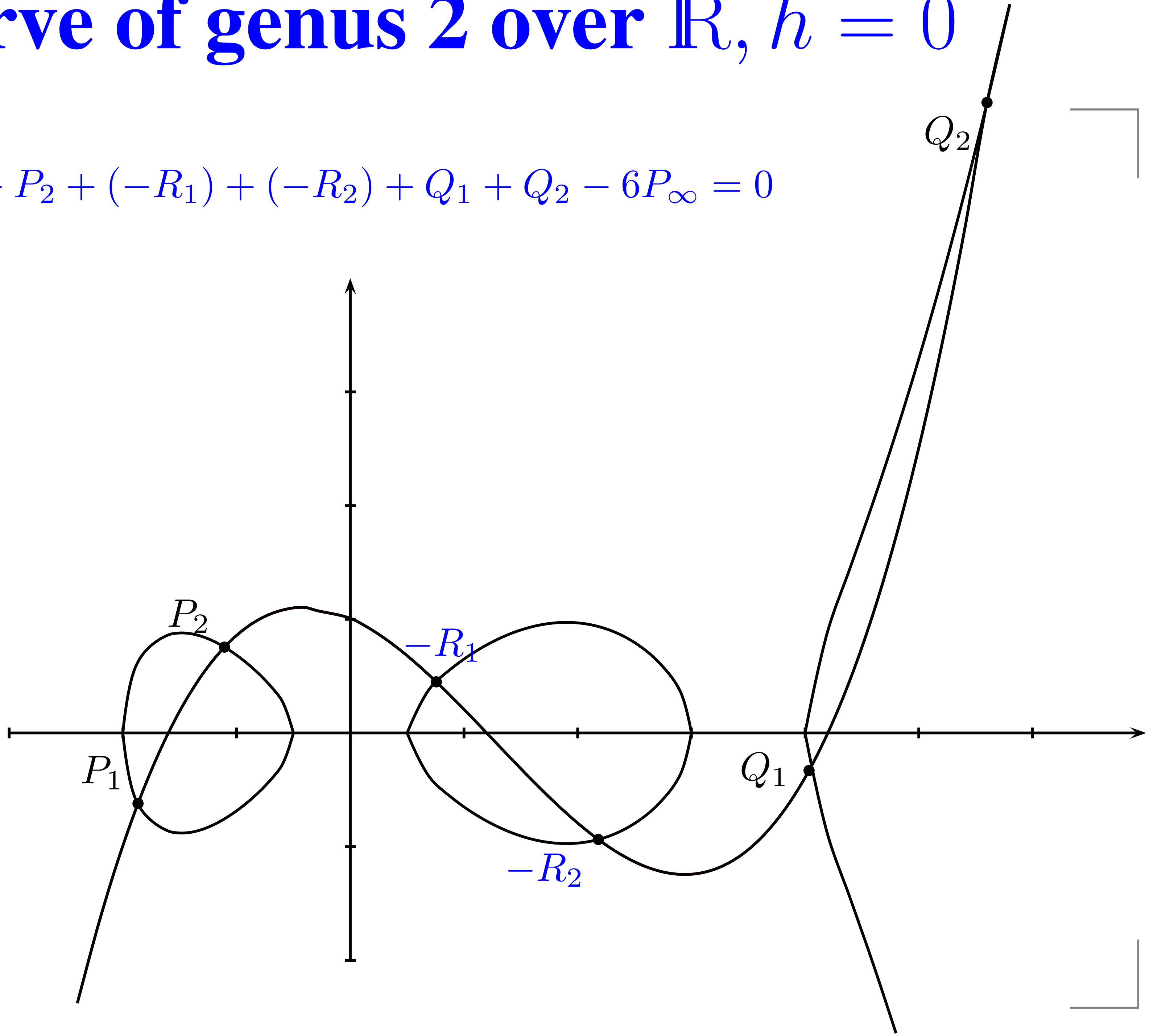
Curve of genus 2 over $\mathbb{R}$, $h = 0$

points on red line $(-6P_\infty)$ form principal divisor



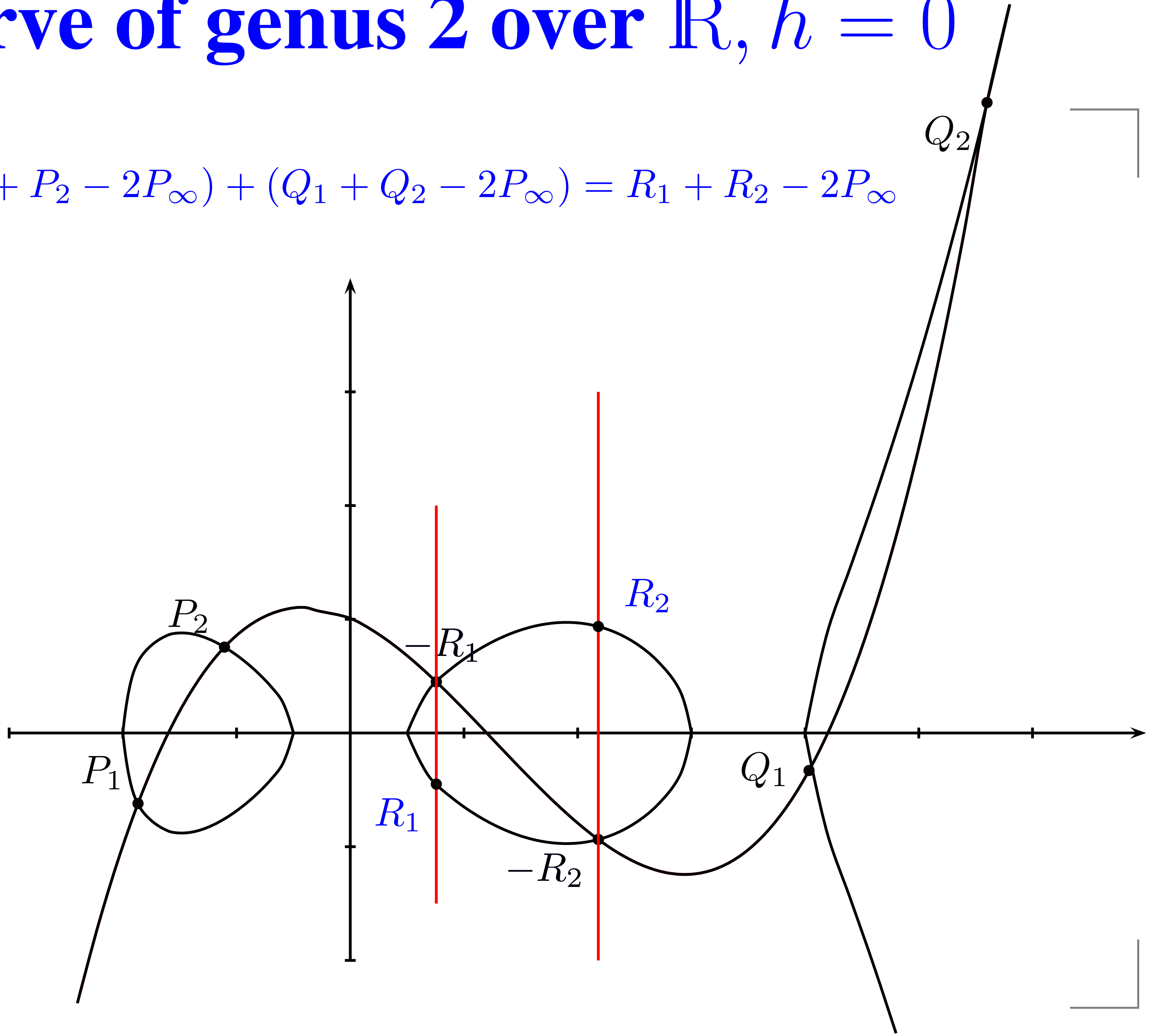
Curve of genus 2 over $\mathbb{R}, h = 0$

$$P_1 + P_2 + (-R_1) + (-R_2) + Q_1 + Q_2 - 6P_\infty = 0$$



Curve of genus 2 over $\mathbb{R}$, $h = 0$

$$(P_1 + P_2 - 2P_\infty) + (Q_1 + Q_2 - 2P_\infty) = R_1 + R_2 - 2P_\infty$$



Number of points

Serre bound:

$$|\#C(\mathbb{F}_q) - q - 1| \leq g\lfloor 2\sqrt{q} \rfloor.$$

Each divisor class has representative with at most g affine points. This motivates the Hasse-Weil bound

$$\#\mathrm{Pic}^0_C(\mathbb{F}_q) = q^g + O(q^{(2g-1)/2}).$$

So a genus 2 curve needs only half the field size.

Mumford Representation

Divisor class group is isomorphic to ideal class group $\mathbf{Cl}(C/\mathbb{F}_q)$ of $\mathbb{F}_q(x,y)/(C)$.
Elements D represented by two polynomials

$$D = [u(x), v(x)]; u, v \in \mathbb{F}_q[x],$$

u monic, $\deg v < \deg u \leq g, \quad u | v^2 + vh - f.$

$\Rightarrow$ compact representation.

Isomorphism is explicit via

$$P_i = (x_i, y_i) \Leftrightarrow u(x_i) = 0, v(x_i) = y_i \text{ with multiplicity}$$

Arithmetic uses *"only"* arithmetic on polynomials ... but is far less efficient than on elliptic curves (if applied directly).

Arithmetic on hyperelliptic curves

Composition & Reduction (Cantor/Koblitz)

`IN:` $D_1 = [u_1, v_1], D_2 = [u_2, v_2], C : y^2 + h(x)y = f(x)$
`OUT:` $D = [u, v]$ reduced with $D \sim D_1 + D_2$

1. compute $d_1 = \gcd(u_1, u_2) = e_1 u_1 + e_2 u_2$
2. compute $d = \gcd(d_1, v_1 + v_2 + h) = c_1 d_1 + c_2(v_1 + v_2 + h)$
3. let $s_1 = c_1 e_1, s_2 = c_1 e_2, s_3 = c_2$
4. $u = \frac{u_1 u_2}{d^2} \quad v = \frac{s_1 u_1 v_2 + s_2 u_2 v_1 + s_3(v_1 v_2 + f)}{d} \bmod u$
5. let $u' = \frac{f - vh - v^2}{u} \quad b' = (-h - v) \bmod u'$
6. if $\deg u' > g$ put $u := u', v := v'$ goto step 5
7. make u monic

Explicit Arithmetic $g = 2$

2000	Harley (odd char.)	$\Rightarrow$ 2 inversions
2001	L. (arbitrary char.)	
2001	Matsuo, Chao, Tsujii (faster)	
2002	Miyamoto, Doi, Matsuo, Chao, Tsujii	$\Rightarrow$ 1 inv.
2002	Takahashi	
2002	L. (arbitrary char.)	
2002	Sugizaki, Matsuo, Chao, Tsujii (even char.)	

2003 Mishra, Sakar Formulae for parallel implementation

2004 L., Stevens $\Rightarrow$ distinction of 3 different cases, optimal affine doubling formulae for each.

Koblitz curves

Koblitz curves (subfield curves)

Hyperelliptic curve

$$C : y^2 + h(x)y = f(x),$$

with $h(x), f(x) \in \mathbb{F}_p[x]$, p **small**,

consider curve over $\mathbb{F}_{p^n}$, n sufficiently large.

Example:

$$C : y^2 + xy = x^5 + x^2 + 1$$

hyperelliptic curve of genus 2 over $\mathbb{F}_2$
hyperelliptic curve of genus 2 over $\mathbb{F}_{2^n}$

History: Koblitz '89; Günther, L., Stein '00; L. 01 (Ph.D.)

Frobenius endomorphism I

- Operates on ideal classes $D = [u(x), v(x)]$ via

$$\sigma(D) = [\sigma(u(x)), \sigma(v(x))],$$

where $\sigma(\sum d_i x^i) = \sum d_i^q x^i$.

- Execution of Frobenius endomorphism is much faster than one group operation.
- σ satisfies characteristic polynomial

$$P(T) = T^{2g} + a_1 T^{2g-1} + \cdots + a_g T^g + \cdots + a_1 p^{g-1} T + p^g$$

with integer coefficients, degree $2g$

Frobenius endomorphism II

- Use σ for faster scalar multiplication by repeatedly replacing

$$p^g(D) = -(\sigma^{2g}(D)+a_1\sigma^{2g-1}(D)+\cdots+a_g\sigma^g(D)+\cdots+a_1p^{g-1}\sigma(D)).$$

- Coefficient set is $\{0, \pm 1, \pm 2, \ldots, \pm\lceil\frac{q^g-1}{2}\rceil\}$
  $\Rightarrow$ use small q only.
- Use of $\sigma^n(D) = D$ ensures Frobenius expansions of length at most n.

Trace Zero Varieties

Trace Zero Subgroup G

Use idea of Koblitz curves with a small extension $n = 3, 5$ (should be prime) and large enough p

Trace Zero Subgroup G:

- $\{D | \sigma^{n-1}(D) + \sigma^{n-2}(D) + \cdots + \sigma(D) + D = 0\}$

Parameter:

- $n \in \{3, 5\}$ for $g = 1$.
- $n = 3$ for $g = 2$

History: Frey '98, Naumann '99 ($g = 1$), Diem '01
L. '01 ($g = 2$), Blady '02 ($g = 1$), (Weimerskirch '02)
Silverberg/Rubin '02 (supersingular curves)
Avanzi, Diem, Frey, L., Scholten '03

Geometric construction I

Example: elliptic curve

$$y^2 = x^3 + Ax + B \text{ over } \mathbb{F}_{p^n} = \mathbb{F}_p[\alpha]$$

write $x = \sum_{i=0}^{n-1} x_i \alpha^i$, $y = \sum_{i=0}^{n-1} y_i \alpha^i$

sort according to powers of α

$$\Rightarrow n\text{-dimensional variety over } \mathbb{F}_p.$$

in general $\Rightarrow ng$-dimensional variety over $\mathbb{F}_p$.

(in mathematical terms: do Weil descent on $\mathbb{F}_p$)

Geometric construction II

Trace condition (broken down to this g-dimensional variety over $\mathbb{F}_p$) cuts out subvariety $\mathcal{G}$

- $\mathcal{G}$ isomorphic to G (as groups)
- $\dim \mathcal{G} = g(n-1)$.
- group size computable from $P(T)$

$$\#\mathcal{G} = \#G \sim \#\mathbf{Cl}(C/\mathbb{F}_{p^n})/\#\mathbf{Cl}(C/\mathbb{F}_p) \sim p^{(n-1)g}$$

Security – Idea

Look for curves over $\mathbb{F}_p$ with Jacobian a subvariety of this $g(n-1)$ dimensional variety $\Rightarrow$ if it has too small genus then this is susceptible to attacks.

Example:
For $g = 1, n = 3$ one could find $J_{\tilde{C}}$ such that $\mathcal{G} \subseteq J_{\tilde{C}}$. By comparing sizes we get genus$(\tilde{C}) \geq 2 \Rightarrow$ security at least p – same as with classical squareroot-attacks (rho, kangaroo, baby-step-giant-step)

Results

- $g = 1, n = 3$: no security risk, as long as genus 2 curves are secure.
- Diem/Scholten 03, building on Howe:
  - $g = 2, n = 3$: one can easily exclude resulting genus $g < 5$, by checking that there are no points of order 3 over $\mathbb{F}_{p^2}$ (the quadratic extension field).
  - $g = 1, n = 5$: same, look for 5-torsion points
  - ($g = 1, n = 7$: family of curves of genus 8, *not* proposed)

Representations $g = 2$

(Interesting from mathematics point of view, necessary for implementations using explicit formulae)

Theorem

Put $g = 2, n = 3$*. Let* $2, 3 \nmid \#\mathbf{Cl}(C/\mathbb{F}_{p^3})$*. The nontrivial elements in the subgroup of order* l *of the trace zero variety* G *are the divisor classes represented by*

$$P_1 + P_2 - 2P_\infty \notin \mathrm{Div}(C/\mathbb{F}_p)^0,$$

where $P_1 \neq P_2, \sigma(P_2), \sigma^2(P_2)$*.*

Scalar multiples in G

Frobenius endomorphism satisfies characteristic polynomial

$P(T) = T^2 + a_1 T + p$ resp. $P(T) = T^4 + a_1 T^3 + a_2 T^2 + a_1 p T + p^2$

and trace relation

$$T^2 + T + 1 \text{ resp. } T^4 + T^3 + T^2 + T + 1.$$

Instead of computing mD for $0 \le m < l$, $l \sim p^{(n-1)g}$ group oder $\langle D \rangle \subseteq G$ use

$$r_0 D + \cdots + r_{n-2} \sigma(D), \ |r_i| \sim \sqrt[n]{m}.$$

and (pairwise) Joint Sparse Form (JSF) of r_i
and/or start with random r_i.

Joint Sparse Form (JSF)

- (JSF 1) Of any three consecutive columns at least one is a zero column.
- (JSF 2) It is never the case that $u_{i,j+1}u_{i,j} = -1$.
- (JSF 3) If $u_{i,j+1}u_{i,j} \neq 0$ then $u_{1-i,j+1} = \pm 1$ and $u_{1-i,j} = 0$.

Example: $k_0 = 403, k_1 = 334$

$$\begin{array}{lcl} k_0 & = & \langle\ 1\ 0\ \text{-}1\ 0\ 0\ 1\ 0\ 1\ 0\ \text{-}1\ \rangle \\ k_1 & = & \langle\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ \text{-}1\ 0\ \rangle \end{array}$$

$$\begin{array}{lcl} k_0 & = & \langle\ 1\ 0\ \text{-}1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ \rangle \\ k_1 & = & \langle\ 1\ 0\ \text{-}1\ \text{-}1\ 0\ 1\ 0\ 0\ \text{-}1\ 0\ \rangle \end{array}$$

How to obtain expansion?

- Use idea of Koblitz curves and expand
  $\Rightarrow$ too much overhead for too large coefficients.
- GLV curves use lattice algorithms for expansion, not applicable for small devices.
- "Naive" expansion and wrapping with trace zero relation works quite well.
  $\Rightarrow$ slightly larger coefficients than optimal
  but very cheap computation,
  only operations modulo p.
- Use random coefficients of appropriate size, avoid collisions.

Avoid collisions I

(necessary when choosing random $r_i, i = 1, \ldots, g-1$)

Theorem

C elliptic curve over $\mathbb{F}_p$, $P(T) = T^2 + a_1 T + p$. D generator of subgroup of prime order l in G for $n = 3$.

$$\mathbf{r} := \min\{\frac{l}{p - a_1}, \frac{p-1}{\gcd(p-1, a_1 - 1)}\}.$$

The $\mathbf{r}^2$ classes $r_0 D + r_1 \sigma(D),\ 0 \leq r_i < \mathbf{r}$ are distinct.

Avoid collisions II

Theorem

C genus 2 hyperelliptic curve over $\mathbb{F}_p$, $P(T) = T^4 + a_1T^3 + a_2T^2 + a_1pT + p^2$. D generator of subgroup of prime order l in G for $n = 3$.

$$\mathbf{r} := \min\left\{\left\lfloor \frac{l}{m} \right\rfloor, \frac{p^2 - a_2 + a_1}{\gcd(p^2 - a_2 + a_1, a_1p - a_2 + 1)}\right\},$$

where $m = \max\{p^2 + a_1p - 2a_2 + a_1 + 1, p^2 + a_1 - a_1p - 1\}$. The $\mathbf{r}^2$ classes $r_0D + r_1\sigma(D),\ 0 \le r_i < \mathbf{r}$ are distinct.

Avoid collisions III

Theorem
C elliptic curve over $\mathbb{F}_p$,
$P(T) = T^2 + a_1 T + p$, D generator of a subgroup of prime order l in G for $n = 5$.

$$\mathbf{r} := \min\left\{\frac{l}{(1+p+|a_1|p)M}, \frac{|p^2 - a_1^2 p + a_1 p - p + 1|}{\gamma}\right\},$$

where

$M = \max\left\{|p^2 - a_1^2 p + 3a_1 p - 2p - a_1^3 + a_1^2 - a_1 + 2|, |p^2 - a_1^2 p - a_1 p + a_1^3 - a_1^2 + a_1|\right\}$

and $\gamma = \gcd(p^2 - a_1^2 p + a_1 p - p + 1,\ 2a_1 p - p - a_1^3 + a_1^2 - a_1 + 1)$.

The $\mathbf{r}^4$ classes $r_0 D + r_1 \sigma(D) + r_2 \sigma^2(D) + r_3 \sigma^3(D),\ 0 \leq r_i < \mathbf{r}$ are distinct.

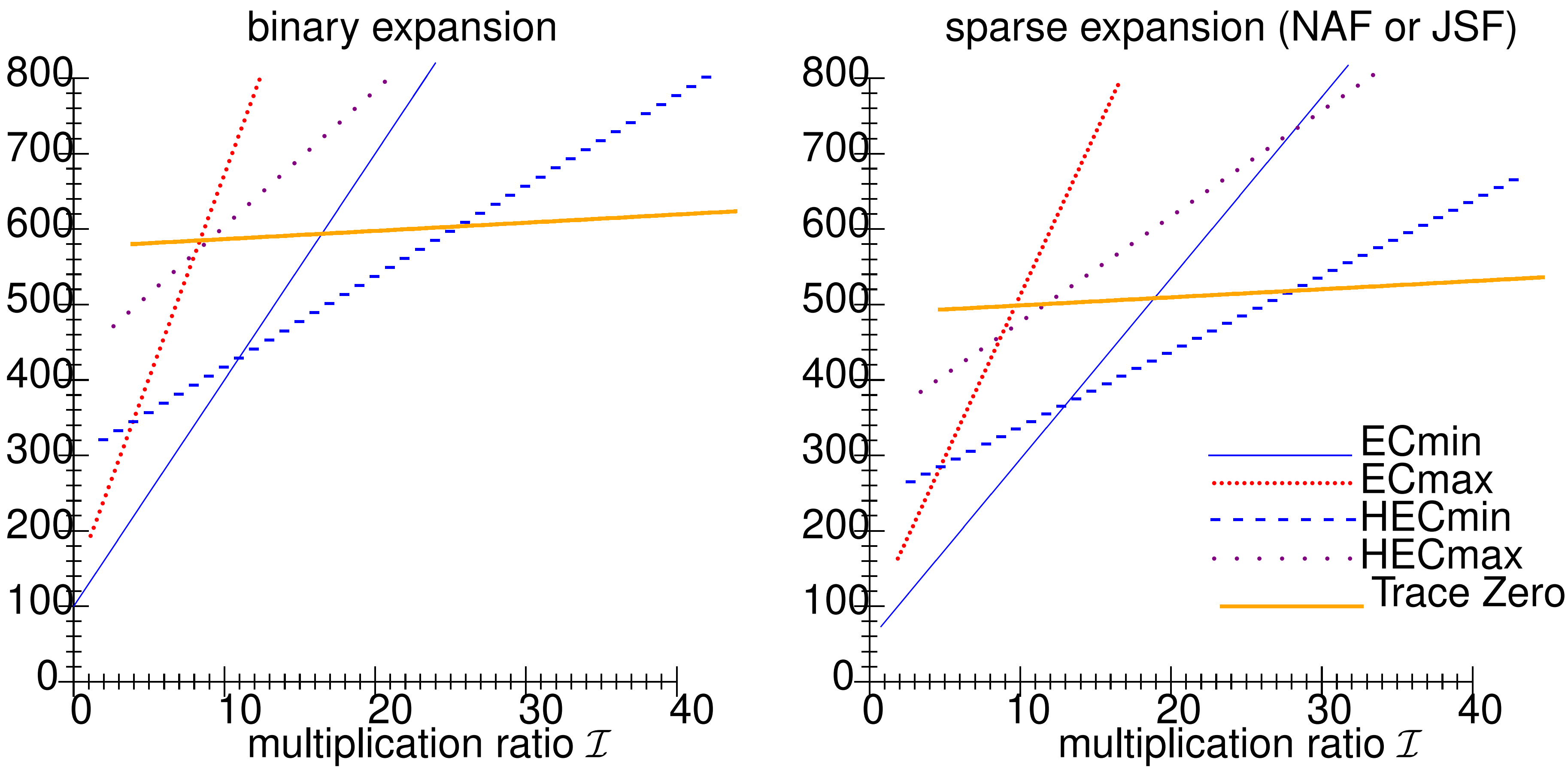
Comparison I, $g = 2, n = 3$

	Elliptic, in $\mathbb{F}_{p''}$			Genus 2, in $\mathbb{F}_{p'}$			G ($g = 2$), in $\mathbb{F}_p$		
	Inv.	Sqr.	Mult.	Inv.	Sqr.	Mult.	Inv.	Sqr.	Mult.
Add.	1	1	2	1	3	22	1	21	141
Doub.	1	2	2	1	5	22	1	33	141
m-fold	6λ	10λ	12λ	6λ	26λ	132λ	3.5λ	97.5λ	501.5λ
NAF/ JSF	4.8λ	8.8λ	9.6λ	4.8λ	22.4λ	105.6λ	3λ	87λ	423λ

using efficient arithmetic in $\mathbb{F}_{p^3}$

(NAF includes same number of precomputations)

Comparison ($g = 2, n = 3$, worst case!)



Implementation – very preliminary ...

(Joint work with R. Avanzi)

bits		$E/\mathbb{F}_{p'}$, $g=1$	$\mathbf{Cl}(C/\mathbb{F}_{p''})$, $g=2$	$G, n=3$, $g=1$	$G, n=5$, $g=1$	$G, n=3$, $g=2$
160	unsig.	1,6ms	1,71ms	1,48ms	1,3ms	3,18ms
	prec.			1,38ms	1,22ms	2,98ms
192	unsig.	2ms	2,1ms	1,94ms	1,76ms	3,7ms
	prec.			1,86ms	1,68ms	3,3ms
256	unsig.	–	4,4ms	4,3ms	2,74ms	5,86ms
	prec.			3,96ms	2,6ms	5,38ms

The End

The END

Thanks!